An approach to computing downward closures

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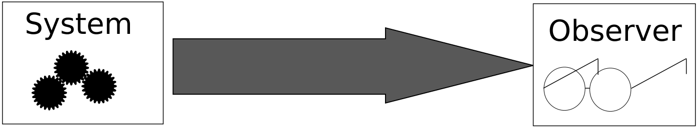
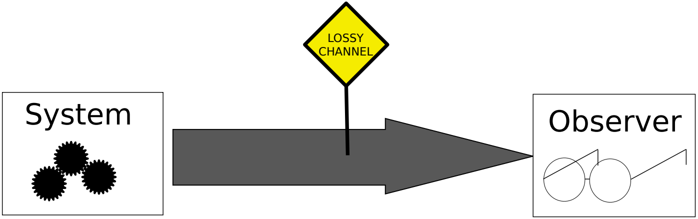
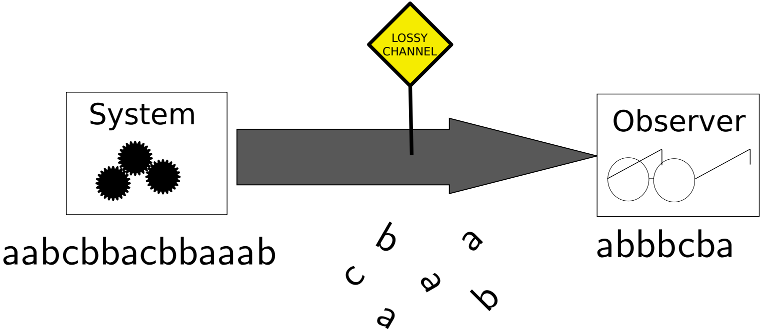

System
Observer


System
LOSSY
CHANNEL
Observer



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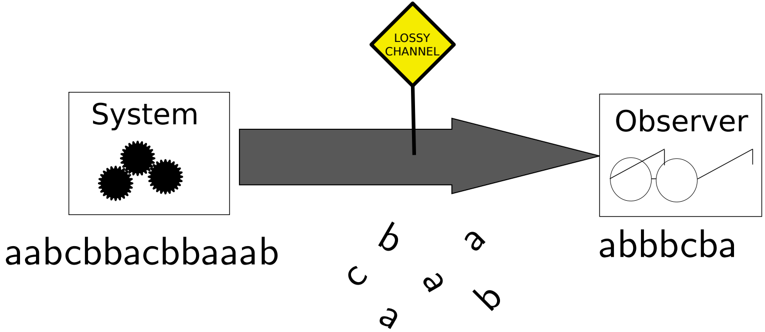

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Downward Closures

- $u \preccurlyeq v$: u is a subsequence of v
- $L{\downarrow} = \{u \in X^* \mid \exists v \in L \colon u \preccurlyeq v\}$
- Observer sees precisely $L{\downarrow}$

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Problem

- Finite automaton for $L\downarrow$ exists for every L.
- How can we compute it?

Negative results

Theorem (Gruber, Holzer, Kutrib 2007)

Downward closures are not computable when infinity or emptiness are undecidable.

Theorem (Mayr 2003)

The reachability set of lossy channel systems is not computable.

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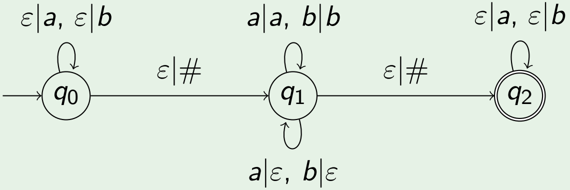
- Weak form of stack nesting
- Adding Counters

A general approach

Example (Transducer)



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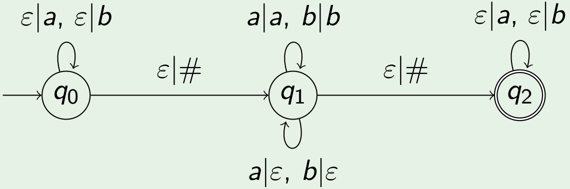

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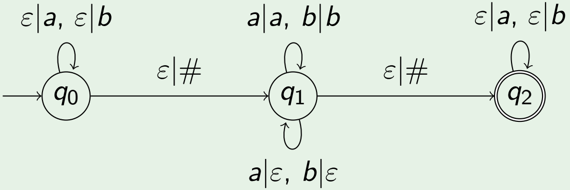

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{y \in X^* \mid \exists x \in L : (x, y) \in T\}$$

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If $\mathcal{C}$ is a full trio, then downward closures are computable for $\mathcal{C}$ if and only if the simultaneous unboundedness problem is decidable:

Given A language $L \subseteq a_1^* \cdots a_n^*$ in $\mathcal{C}$

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Equivalently, we check whether it is true that:

for each $k \geqslant 0$, there are $x_1, \ldots, x_n \geqslant k$ with $a_1^{x_1} \cdots a_n^{x_n} \in L$

Theorem (Jullien 1969, Abdulla et. al. 2004)

Every language $L\downarrow$ can be written as a finite union of sets of the form

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.

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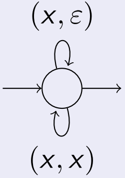

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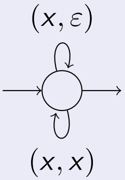

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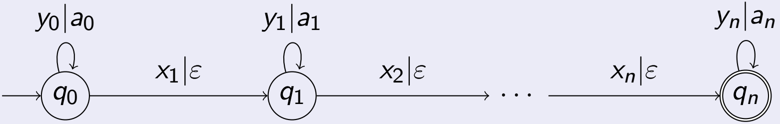

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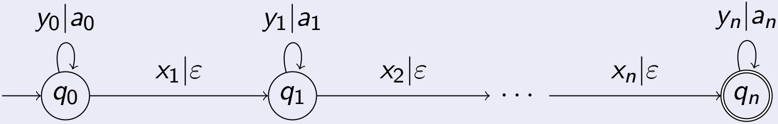

y_i: word containing each letter of Y_i once. Then:

$$T(L{\downarrow}){\downarrow} = a_0^*\cdots a_n^* \quad \text{iff} \quad Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$$

New algorithm for every known positive case

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(Generalize 0L-systems)

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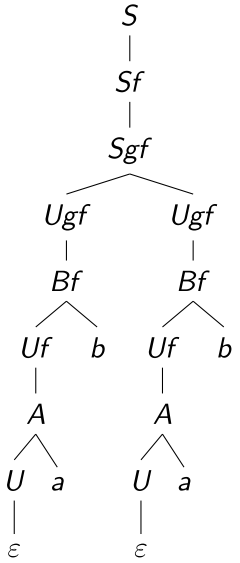

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Then, $a_1^* \cdots a_n^* \subseteq L(G){\downarrow}$ iff $a_1^* \cdots a_n^* \subseteq L(G_D){\downarrow}$ for some D.

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Indirect symbols: $\{a_3, a_4, a_9\}$

Goal: bound nonterminal occurrences

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Idea

Instead of unfolding a_i-subtree with root Au, $u \in I^*$, apply transducer to u

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Instead of unfolding a_i-subtree with root Au, $u \in I^*$, apply transducer to u
However: Precise simulation not possible

Preserving $a_1^* \cdots a_n^* \subseteq L(G)\downarrow$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N} \cup \{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (Au, v) \in T\}$$
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Proposition

For each indexed grammar G, one can construct a rational transduction T with $f_T \approx f_G$.

$f \approx g$: f is unbounded on the same subsets as g
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$\Rightarrow$ Bound on nonterminal occurrences, "breadth-bounded"

Remaining problem

- Given: Breadth-bounded indexed grammar G, $L(G) \subseteq a_1^* \cdots a_n^*$
- Is $a_1^* \cdots a_n^*$ included in $L(G)\downarrow$?

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Proposition

Breadth-bounded indexed grammars have effectively semilinear Parikh images.

Thank you for your attention!