On Boolean closed full trios and rational Kripke frames

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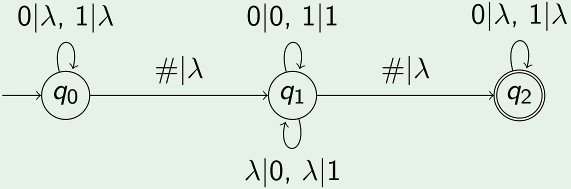
Example (Transducer)



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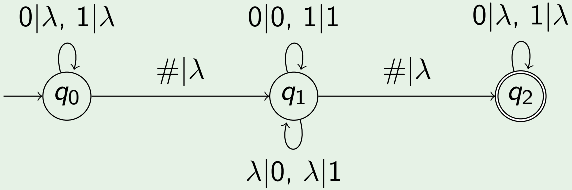

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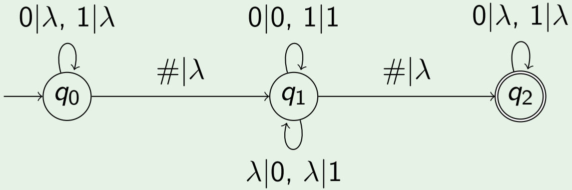

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{x \in X^* \mid \exists y \in L : (x, y) \in T\}$$

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Boolean closed full trios

Are there Boolean closed full trios beyond REG?

- Automatic structures beyond regular languages
- Complementation closure for union closed full trios

$\mathrm{RE}(\mathcal{C})$: Accepted by Turing machine with oracle $L \in \mathcal{C}$.

Definition

Arithmetical hierarchy:

$$\Sigma_0 = \mathrm{REC}, \qquad \Sigma_{n+1} = \mathrm{RE}(\Sigma_n) \text{ for } n \geqslant 0, \qquad \mathrm{AH} = \bigcup_{n \geqslant 0} \Sigma_n.$$

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Theorem

Let $\mathcal{T}$ be a Boolean closed full trio. If $\mathcal{T}$ contains any non-regular language L, then $\mathcal{T}$ includes $\mathsf{AH}(L)$.

Proof I

Let $\Delta = \{+, -, z\}$: increment, decrement, and zero test.

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$u \equiv_L v$: for each $w \in X^*$, $uw \in L$ iff $vw \in L$.

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Theorem (Myhill-Nerode)

L is regular if and only if $\equiv_L$ has finite index.

Proof II

Idea: In order to obtain C, construct $\hat{C}$:

Definition

Let $\hat{C}$ (*counter*) be the set of all words

$$v_0\delta_1 v_1 \cdots \delta_m v_m \# u_0 \# \cdots u_n \#$$

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- if $\delta_i = z$, then $v_{i-1} \equiv_L v_i \equiv_L u_j \equiv_L u_0$.

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Since L is non-regular, C can be obtained from $\hat{C}$.

Proof III

$$W_1 = \{u\#v\#w \mid u, v, w \in X^*,\ uw \in L\},$$
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$$= (X^*\#)^* \backslash \{ru\#sv\#t \mid r, s, t \in (X^*\#)^*, u\#v \in P\}.$$

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$$\begin{aligned} M = \{v_1{+}v_2\#u_1\#u_2 \mid v_1\#u_1 \in P,\ v_2\#u_2 \in P\} \\ \cup \{v_1{-}v_2\#u_1\#u_2 \mid v_1\#u_2 \in P,\ v_2\#u_1 \in P\} \\ \cup \{v_1 z v_2\#u_1\#u_2 \mid v_1\#v_2 \in P,\ v_1\#u_1 \in P,\ u_2 \in X^*\} \end{aligned}$$

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Let E (*error*) be the set of words $v_1 \delta v_2 \# u_0 \# \cdots u_n \#$ such that for every $1 \leqslant j \leqslant n$, we have $v_1 \delta v_2 \# u_{j-1} \# u_j \notin M$ or we have $\delta = z$ and $v_1 \not\equiv_L u_0$.

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$$E' = \{v_1 \delta v_2 \# r u_1 \# u_2 \# s \mid v_1 \delta v_2 \# u_1 \# u_2 \in M, r, s \in (X^* \#)^*\}$$

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$$\begin{aligned} M = {} & \{v_1{+}v_2\#u_1\#u_2 \mid v_1\#u_1 \in P,\ v_2\#u_2 \in P\} \\ & \cup \{v_1{-}v_2\#u_1\#u_2 \mid v_1\#u_2 \in P,\ v_2\#u_1 \in P\} \\ & \cup \{v_1 z v_2\#u_1\#u_2 \mid v_1\#v_2 \in P,\ v_1\#u_1 \in P,\ u_2 \in X^*\} \end{aligned}$$

Let E (*error*) be the set of words $v_1 \delta v_2 \# u_0 \# \cdots u_n \#$ such that for every $1 \leqslant j \leqslant n$, we have $v_1 \delta v_2 \# u_{j-1} \# u_j \notin M$ or we have $\delta = z$ and $v_1 \not\equiv_L u_0$.

$$E' = \{v_1 \delta v_2 \# r u_1 \# u_2 \# s \mid v_1 \delta v_2 \# u_1 \# u_2 \in M, r, s \in (X^* \#)^*\}$$

$$E = \left[(X^* \Delta X^* \#(X^* \#)^* \setminus E'\right]$$

Proof IV

Let M (*matching*) be the set of all words $v_1 \delta v_2 \# u_1 \# u_2$, $v_1, v_2, u_1, u_2 \in X^*$, with

- if $\delta = +$, then $v_1 \equiv_L u_1$ and $v_2 \equiv_L u_2$,
- if $\delta = -$, then $v_1 \equiv_L u_2$ and $v_2 \equiv_L u_1$, and
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Proof V

Let N (*no error*) be the set of words $v_0\delta_1 v_1 \cdots \delta_m v_m \# u_0 \# \cdots u_n \#$ such that for every $1 \leqslant i \leqslant m$, there is a $1 \leqslant j \leqslant n$ with $v_{i-1}\delta_i v_i \# u_{j-1} \# u_j \in M$ and if $\delta_i = z$, then $v_{i-1} \equiv_L u_0$.

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Now we have
$$\hat{C} = N \cap (X^*\Delta)^* X^* \# S.$$

Hence, $C \in \mathcal{T}$.

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For AH$(L) \subseteq \mathcal{T}$: show that $K \in \mathcal{T}$ implies RE$(K) \subseteq \mathcal{T}$ (as above).

Corollary

Let L be non-regular. The smallest Boolean closed full trio containing L is $\mathsf{AH}(L)$.

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Other than the regular languages, no principal full trio is complementation closed.

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Let $\mathcal{T}$ be generated by L. It consists of RL for rational transductions R. Hence, $\mathcal{T}$ is union-closed and $\mathcal{T} \subseteq \mathsf{RE}(L) \subsetneq \mathsf{AH}(L)$. If $\mathcal{T}$ were complementation closed, it would contain $\mathsf{AH}(L)$, contradiction! □

Corollary

Let M be a finitely generated monoid. The following are equivalent:

1. $\mathsf{VA}(M)$ *is complementation closed.*
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If M is finitely generated, $\mathsf{VA}(M)$ is a principal full trio.

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If M is finitely generated, $\mathsf{VA}(M)$ is a principal full trio. Equivalence of 2 and 3 has been shown by Render (2010) and Z. (2011). □

An application

Syntax of multimodal logic

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond_a \varphi \mid \Box_a \varphi$$

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Semantics of multimodal logic

A *Kripke structure* is a tuple

$$\mathcal{K} = (V, (E_a)_{a \in A}, (U_p)_{p \in P}),$$

where

- V is a set of worlds,
- A and P are finite sets of actions and propositions, respectively,
- for every $a \in A$, $E_a \subseteq V \times V$, and
- for every $p \in P$, $U_p \subseteq V$.

The tuple $\mathcal{F} = (V, (E_a)_{a \in A})$ is then also called a *Kripke frame*.

An application

Semantics

For $\mathcal{K} = (V, (E_a)_{a\in A}, (U_p)_{p\in P})$, we have

$$\begin{aligned}
[\![p]\!]_{\mathcal{K}} &= U_p, \\
[\![\neg\varphi]\!]_{\mathcal{K}} &= V \setminus [\![\varphi]\!]_{\mathcal{K}}, \\
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[\![\Box_a\varphi]\!]_{\mathcal{K}} &= \{v \in V \mid \forall u \in V : (v, u) \in E_a \to u \in [\![\varphi]\!]_{\mathcal{K}}\}, \\
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Rational Kripke frames

$\mathcal{F} = (V, (E_a)_{a \in A})$ is called *rational*, if

- $V = X^*$ for some alphabet X
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Theorem (Bekker, Goranko 2007)

If $\mathcal{K} = (V, (E_a)_{a \in A}, (U_p)_{p \in P})$ is rational If $\mathcal{F}$ is rational and U_p is regular for each $p \in P$, the set $[\![\varphi]\!]_{\mathcal{K}}$ is effectively regular. Hence, the model-checking problem is decidable.

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Theorem

Let $X = \{0, 1\}$. There is a rational Kripke frame $\mathcal{F} = (X^*, R, S, T)$, $R, S, T \subseteq X^* \times X^*$ such that for any non-regular L, in the Kripke structure $\mathcal{K} = (X^*, R, S, T, L)$, for each $K \in \mathrm{AH}(L)$, there is a φ with $[\![\varphi]\!]_{\mathcal{K}} = K$.