Recent advances on valence automata as a generalization of automata with storage

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Technische Universität Kaiserslautern

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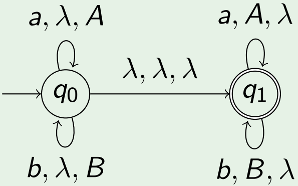
Example (Pushdown automaton)



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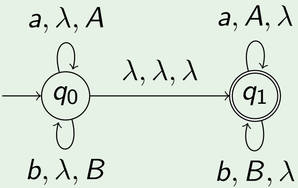

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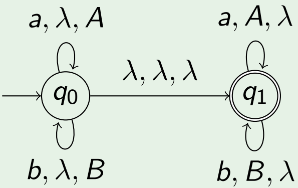

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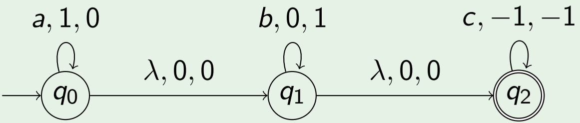

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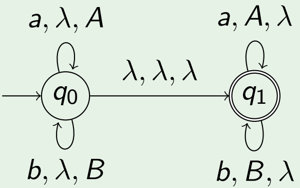

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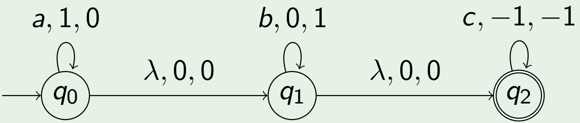

$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

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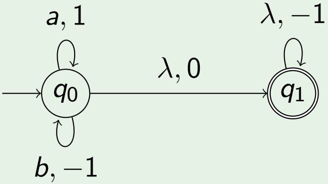

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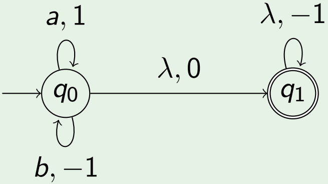

$$L = \{w \in \{a,b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for any prefix } p \text{ of } w\}$$

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

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Each storage mechanism consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \to S$

Model	States	Operations
Pushdown automata	$S = \Gamma^*$	$\mathrm{push}_a : w \mapsto wa,\ a \in \Gamma$ $\mathrm{pop}_a : wa \mapsto w,\ a \in \Gamma$

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Observation

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \text{id}$$

Definition

A *monoid* is

- a set M together with
- an associative binary operation $\cdot : M \times M \to M$ and
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Storage mechanisms as monoids

- Let S be a set of states and $\alpha_1, \ldots, \alpha_n : S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid M.
- The identity map is the neutral element of M.
- M is a decription of the storage mechanism.

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Common generalization: Valence Automata

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Language class

$\mathrm{VA}(M)$ languages accepted by valence automata over M.

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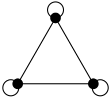
Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^* / \{a\bar{a} = 1\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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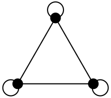

$\mathbb{Z}^3$

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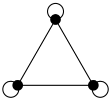

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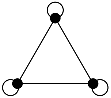

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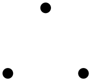

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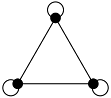

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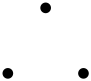

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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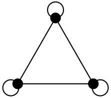

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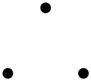

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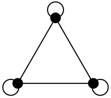

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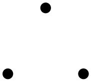

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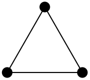

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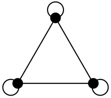

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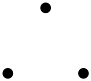

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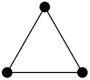

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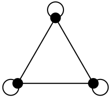

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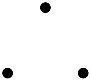

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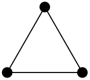

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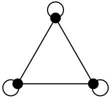

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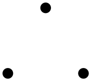

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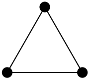

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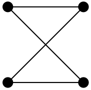

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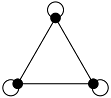

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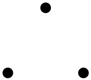

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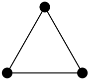

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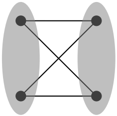

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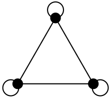

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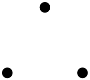

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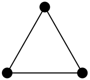

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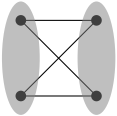

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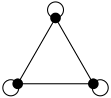

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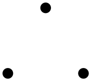

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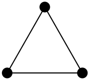

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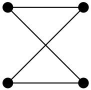
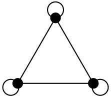


$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

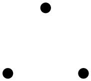
Infinite tape (TM)

Examples



$\mathbb{Z}^3$

Blind multicounter



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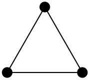

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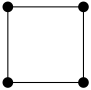

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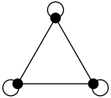

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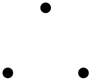

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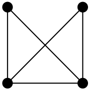

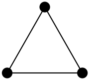

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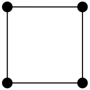

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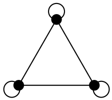

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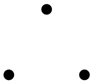

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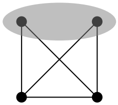

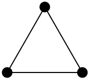

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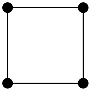

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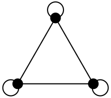

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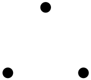

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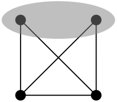

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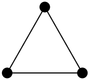

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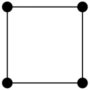

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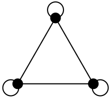

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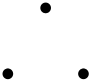

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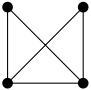

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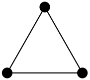
Pushdown + partially blind counters



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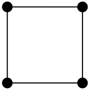

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Silent Transitions

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- When can silent transitions be eliminated?
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For which storage mechanisms can we avoid silent transitions?

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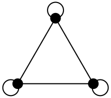
Question

For which storage mechanisms can we avoid silent transitions?

Known so far

- Pushdown automata (Greibach 1965)
- Blind counter automata (Greibach 1978)
- Partially blind counter automata (Greibach 1978 / Jantzen 1979)

Examples, again



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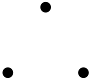

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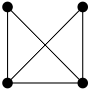

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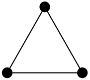
Pushdown + partially blind counters



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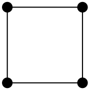

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Infinite tape (TM)

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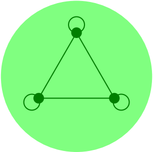

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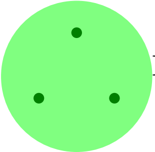

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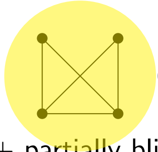

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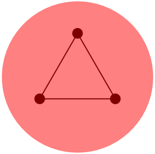

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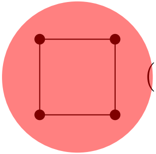

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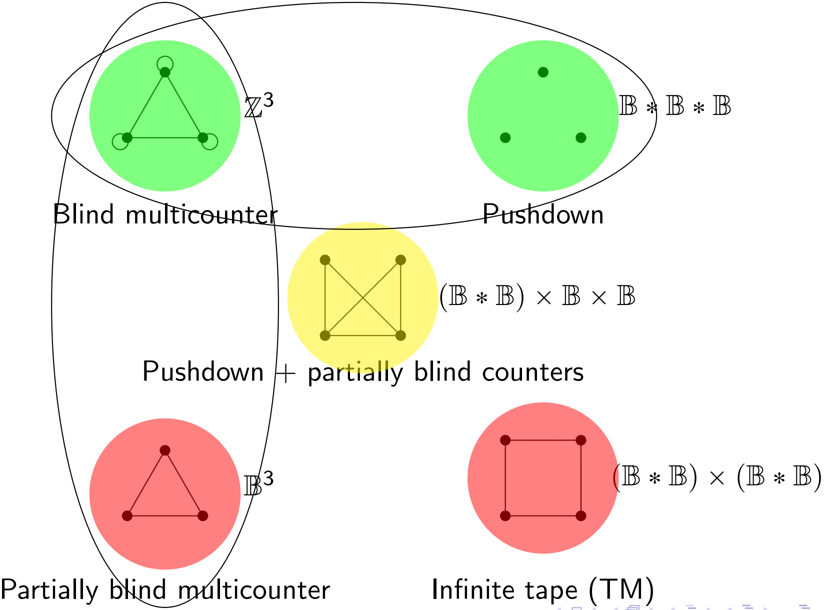

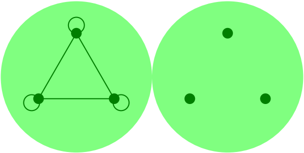

Theorem (Z., ICALP 2013)

Let Γ be a graph such that

- *any two looped vertices are adjacent,*
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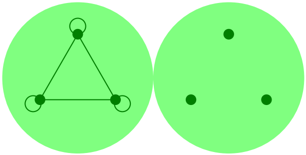

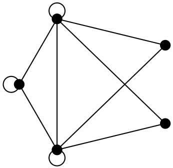

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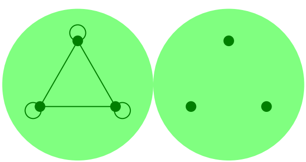

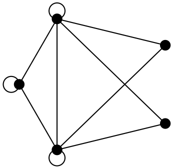

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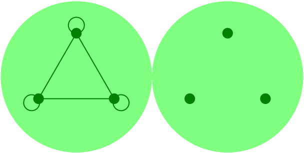

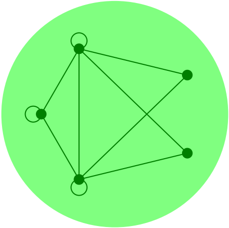

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Positive case

Definition

Let $\mathcal{C}$ be the smallest class of monoids such that

- $1 \in \mathcal{C}$
- if $M \in \mathcal{C}$, then $M \times \mathbb{Z} \in \mathcal{C}$
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Lemma

Let Γ be a graph such that

- *any two looped vertices are adjacent,*
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-  *does not appear as an induced subgraph.*

Then, $\mathbb{M}\Gamma \in \mathcal{C}$.

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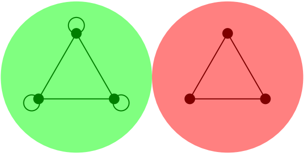
Interpretation of $\mathcal{C}$

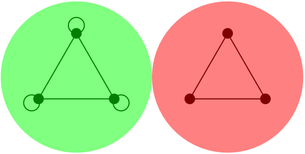
$\mathcal{C}$ corresponds to the class of storage mechanisms obtained by

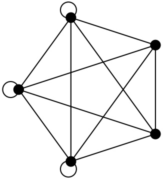
- adding a blind counter ($M \times \mathbb{Z}$) and
- building stacks ($M * \mathbb{B}$).



Theorem (Z., ICALP 2013)

Let Γ be a graph such that between any two distinct vertices, there is an edge.





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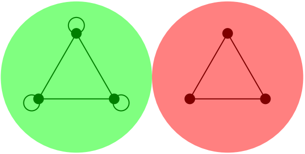

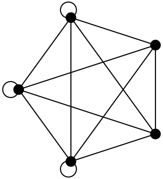

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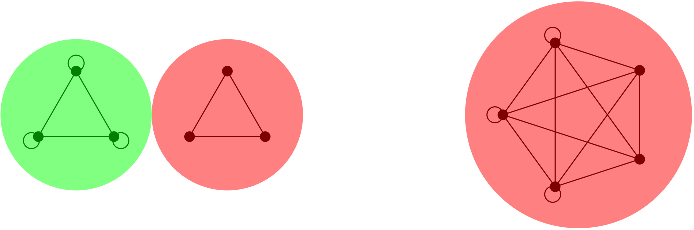
Let Γ be a graph such that between any two distinct vertices, there is an edge. Then $\mathsf{VA}(\mathbb{M}\Gamma) = \mathsf{VA}^+(\mathbb{M}\Gamma)$ *if and only if the number of unlooped nodes is* $\leqslant 1$.



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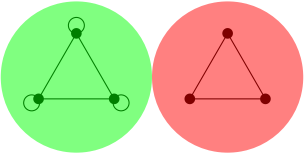

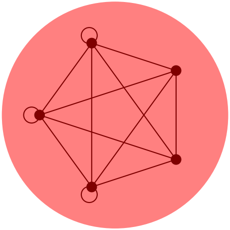

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$$\mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \quad \textit{iff} \quad r \leqslant 1.$$

Observation

$\mathsf{VA}(\mathbb{B} \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B} \times \mathbb{Z}^s)$ already follows from the first theorem.

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Proving $\mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \subsetneq \mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r \geqslant 2$

- Use Greibach's and Jantzen's language

$$L_1 = \{wc^n \mid w \in \{0,1\}^*, n \leqslant \mathrm{bin}(w)\},$$

$$\mathrm{bin}(v0) = 2 \cdot \mathrm{bin}(v), \quad \mathrm{bin}(v1) = 2 \cdot \mathrm{bin}(v) + 1, \quad \mathrm{bin}(\lambda) = 0.$$

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- Known that $L_1 \in \mathsf{VA}(\mathbb{B}^2)$.
- Show $L_1 \notin \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r, s \in \mathbb{N}$.

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Proving $\mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \subsetneq \mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r \geqslant 2$

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- Languages in $\mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s)$ have polynomially many fooling sets
- L_1 has exponential number of fooling sets

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For which monoids M are all languages in $\mathsf{VA}(M)$ semilinear?

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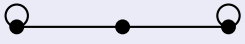
All languages in $\mathsf{VA}(\mathbb{M}\Gamma)$ *are semilinear if and only if*

1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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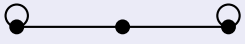
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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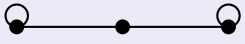
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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- For 4 forbidden induced subgraphs, non-semilinear languages from Petri net and trace theory
- $\mathsf{VA}(\mathbb{B}) \subseteq \mathsf{CF}$
- $M \mapsto M \times \mathbb{Z}$, $(M, M') \mapsto M * M'$ preserve semilinearity

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