Downward closures and complexity

Georg Zetzsche[1]

LSV, CNRS & ENS Cachan
Université Paris-Saclay

Higher-Order Model Checking
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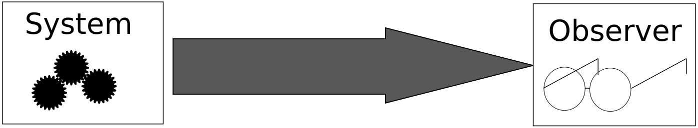
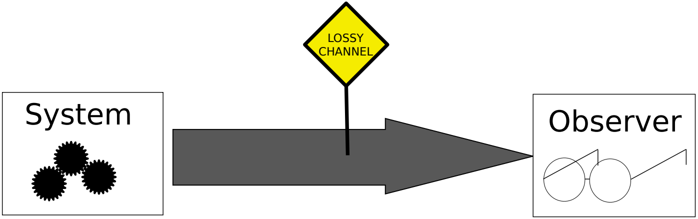

System
Observer


LOSSY
CHANNEL
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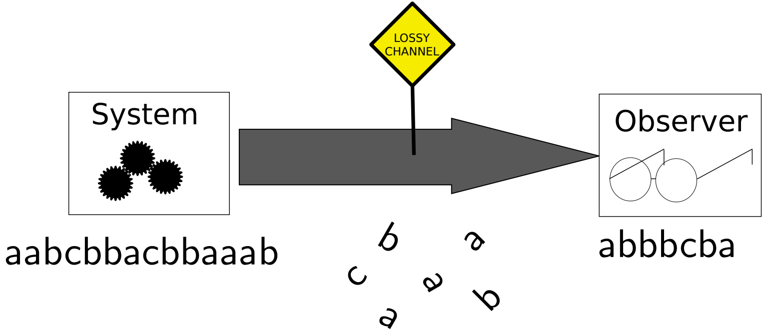

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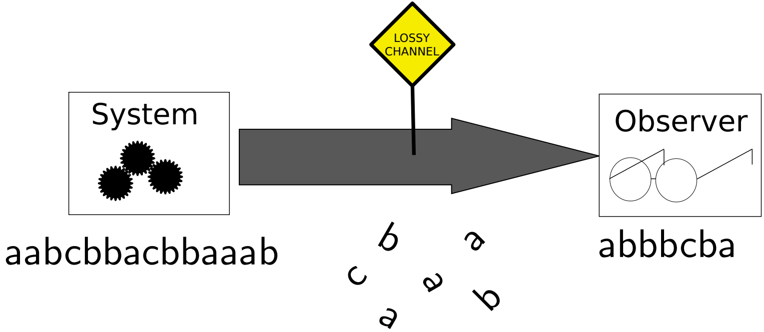

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Downward Closures

- $u \preccurlyeq v$: u is a subsequence of v
- $L{\downarrow} = \{u \in X^* \mid \exists v \in L\colon u \preccurlyeq v\}$
- Observer sees precisely $L{\downarrow}$

Downward Closures

Theorem (Higman/Haines)

For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.

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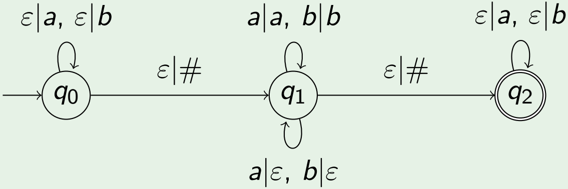
Problem

- Finite automaton for $L{\downarrow}$ exists for every L.
- How can we compute it?

A general approach

Example (Transducer)



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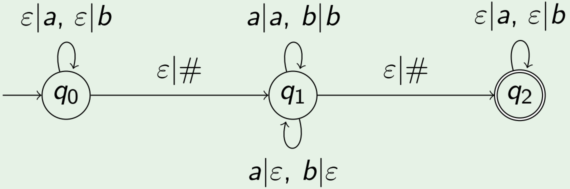

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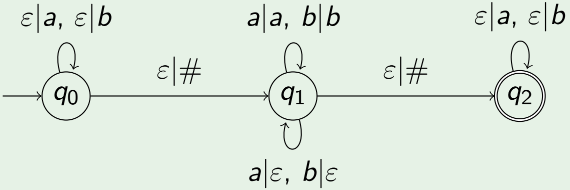

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{y \in Y^* \mid \exists x \in L : (x, y) \in T\}$$

Definition

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Theorem (Z., ICALP 2015)

If $\mathcal{C}$ is a full trio, then downward closures are computable for $\mathcal{C}$ if and only if the simultaneous unboundedness problem is decidable:

Given A language $L \subseteq a_1^* \cdots a_n^*$ in $\mathcal{C}$

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Equivalently, we check whether it is true that:

$$\text{for each } k \geqslant 0, \text{ there are } x_1, \ldots, x_n \geqslant k \text{ with } a_1^{x_1} \cdots a_n^{x_n} \in L$$

Theorem (Jullien 1969, Abdulla et. al. 2004)

Every language $L\!\downarrow$ can be written as a finite union of ideals:

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.

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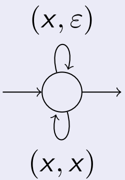

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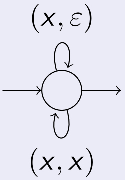

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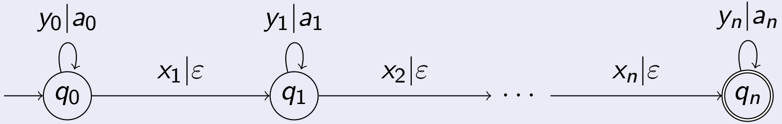

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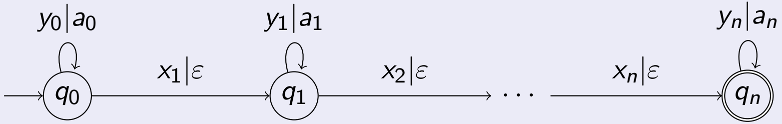

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$$T(L{\downarrow}){\downarrow} = a_0^*\cdots a_n^* \quad \text{iff} \quad Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$$

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- Igor's talk: higher-order recursion schemes

Higher-Order Pushdown Automata

Let Γ be a stack alphabet.
S_n^Γ is the set of order-n stacks:

$$S_0^\Gamma = \Gamma \qquad S_{k+1}^\Gamma = \{[s_1 \cdots s_m]_{k+1} \mid s_1, \ldots, s_m \in S_k^\Gamma\}.$$

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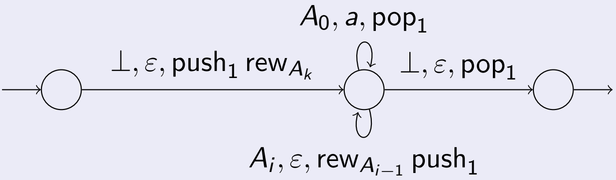

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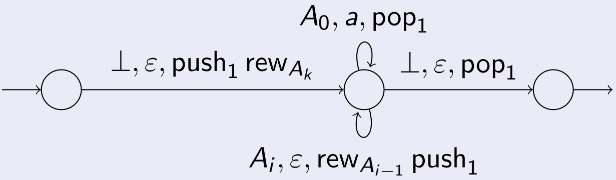

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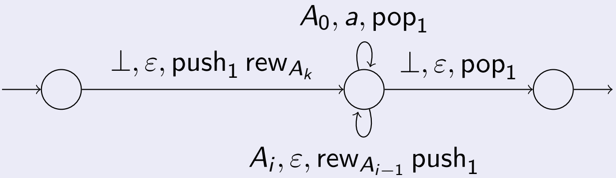

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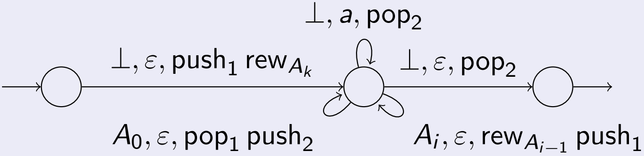

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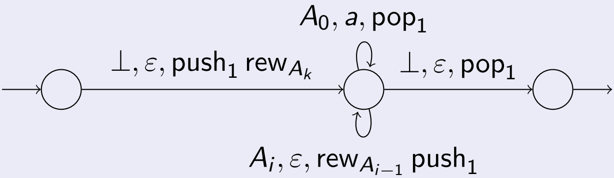

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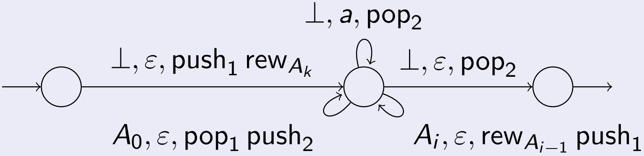

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Given: Language K, L described in $\mathcal{M}$ and $\mathcal{N}$, respectively.

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Proposition (Bachmeier, Luttenberger, Schlund 2015)

If $\mathcal{A}$ is an NFA and $K{\downarrow} \nsubseteq L(\mathcal{A}){\downarrow}$, then there exists a $w \in K{\downarrow} \setminus L(\mathcal{A}){\downarrow}$ with $|w| \leqslant |\mathcal{A}| + 1$.

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Proposition (Bachmeier, Luttenberger, Schlund 2015)

If $\mathcal{A}$ is an NFA and $K{\downarrow} \nsubseteq L(\mathcal{A}){\downarrow}$, then there exists a $w \in K{\downarrow} \setminus L(\mathcal{A}){\downarrow}$ with $|w| \leqslant |\mathcal{A}| + 1$.

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- For an input word $w = x_1 \cdots x_n$, consider the sets Q_i of words reachable by $x_1 \cdots x_i$. Then $Q_0 \supseteq Q_1 \supseteq \cdots$
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$$|L| = \min\left\{\max_j |I_j| \quad : \quad L{\downarrow} = I_1 \cup \cdots \cup I_k \text{ for ideals } I_1, \ldots, I_k\right\}$$

Pumping

Putting a bound on $|L|$ amounts to proving a pumping lemma:
$|L| \leqslant m$ if and only if for every $w \in L$, there is an ideal I such that $|I| \leqslant m$ and $w \in I \subseteq L{\downarrow}$.

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Let $I = Y_0^*\{x_1, \varepsilon\}Y_1^* \cdots \{x_n, \varepsilon\}Y_n^*$. Then the following are equivalent:

1. $I \subseteq L\downarrow$.
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Strategy for $K{\downarrow} \subseteq L{\downarrow}$

- Suppose $|K|$ is polynomial and $|L|$ exponential.
- Guess an ideal I of length $\leqslant |K|$ and verify $I \subseteq K$, $I \nsubseteq L$.
- Represent witness above succinctly.

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- Each DFA has to repeat a state, hence cycle on the last letter of w

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Under mild conditions on the models $\mathcal{M}$ and $\mathcal{N}$: Suppose for each n we have a description of $\{a^{t(n)}\}$ in $\mathcal{M}$ and $\mathcal{N}$ of polynomial size. Then $\mathcal{M} \subseteq_{\downarrow} \mathcal{N}$ is hard for $\mathsf{coNTIME}(t)$.

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Corollary

The problem $\mathrm{HOPA}_n \subseteq_{\downarrow} \mathrm{HOPA}_n$ *is* co-n-NEXP*-hard.*

	Ideal	NFA	OCA	$RBC_{k,r}$	CFG	RBC
Ideal	$\in$ L	NL	NL	NL	P	NP
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