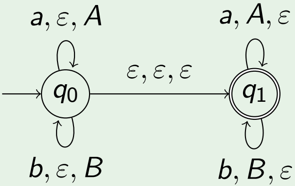
Monoids as Storage Mechanisms

Georg Zetzsche

LSV Cachan

INFINI Group Seminar

Example (Pushdown automaton)



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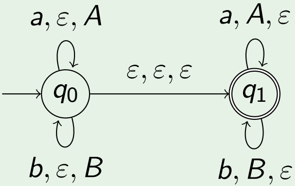

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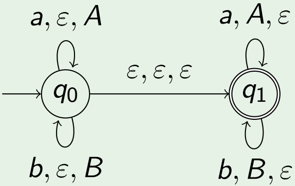

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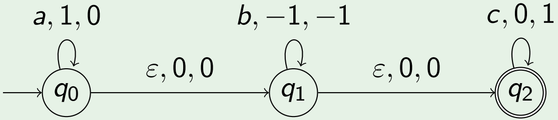

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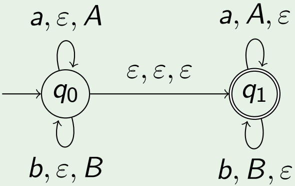

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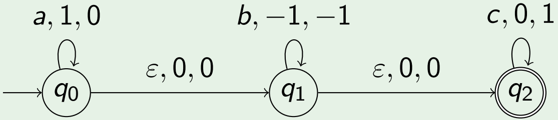
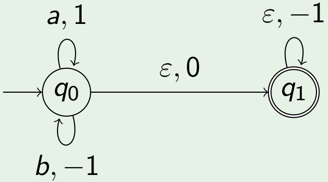


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



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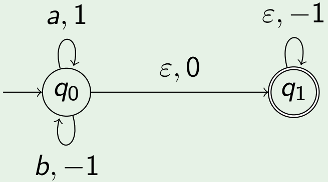

$$L = \{w \in \{a,b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for each prefix } p \text{ of } w\}$$

Storage mechanisms

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

Goal: General insights

Structure of storage $\Leftrightarrow$ computational properties

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Framework

Abstract model with storage as parameter

Valence automata

Definition

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- an associative binary operation $\cdot\colon M \times M \to M$ and
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Language class

VA(M) languages accepted by valence automata over M.

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Questions

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- For which do we have a particular closure property?
- How is the complexity of decision problems affected?
- For which can we compute abstractions?

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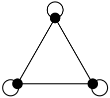
Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^* / \{a\bar{a} = \varepsilon\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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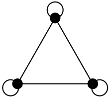

$\mathbb{Z}^3$

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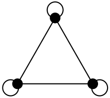

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Blind counter

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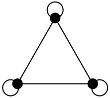

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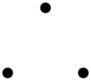

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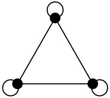

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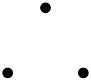

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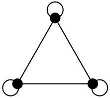

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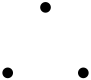

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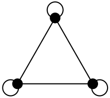

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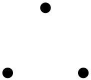

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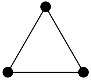

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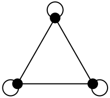

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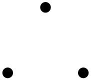

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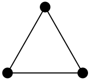

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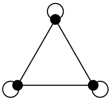

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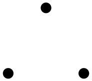

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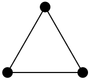

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Partially blind counter

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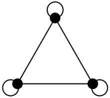

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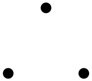

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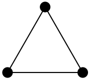

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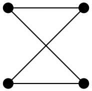

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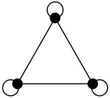

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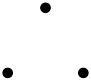

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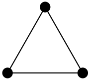

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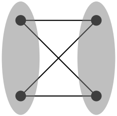

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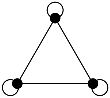

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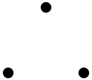

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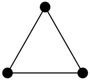

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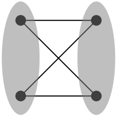

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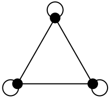

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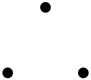

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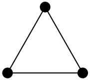

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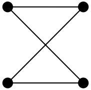

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Infinite tape (TM)

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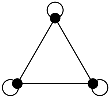

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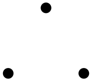

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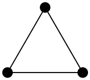

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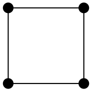

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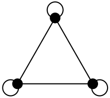

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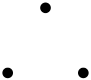

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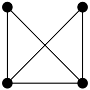

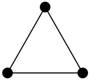

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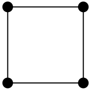
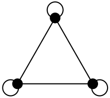


$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

Infinite tape (TM)

Examples



$\mathbb{Z}^3$

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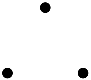
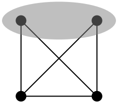
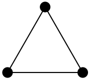


$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown





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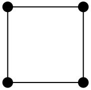

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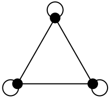

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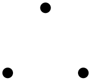

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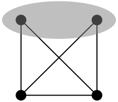

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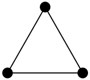

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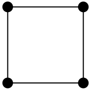

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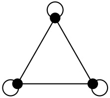

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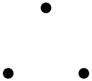

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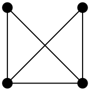

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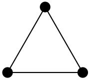
Pushdown + partially blind counters



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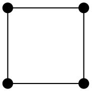

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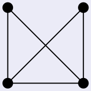

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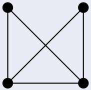
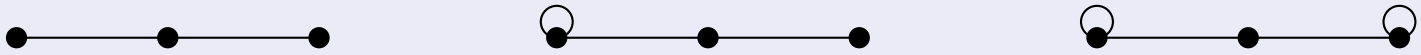


Pushdown + partially blind counters

Decidability a long-standing open problem

Simplest graphs for pushdown + counters



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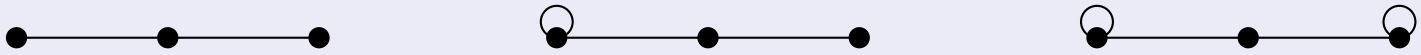

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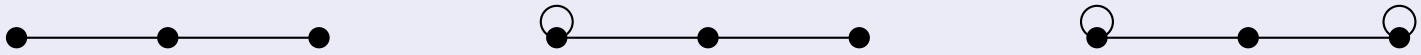

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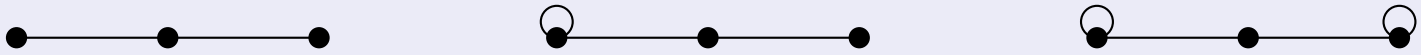

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Let Γ *be PPN-free. Then the following are equivalent:*

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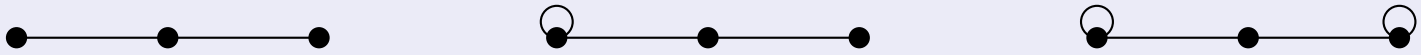

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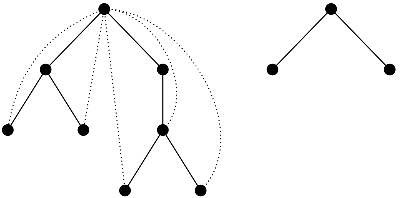
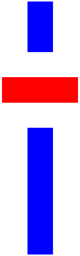
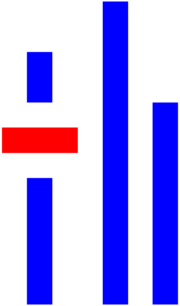
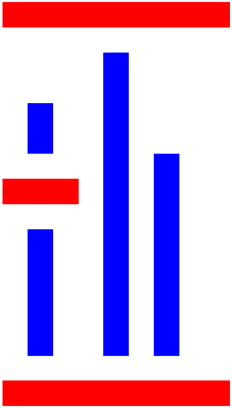
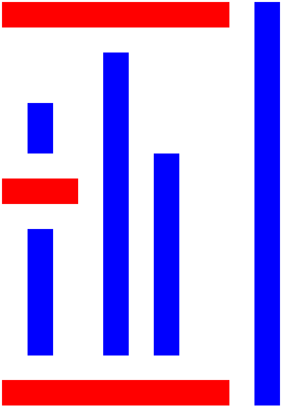
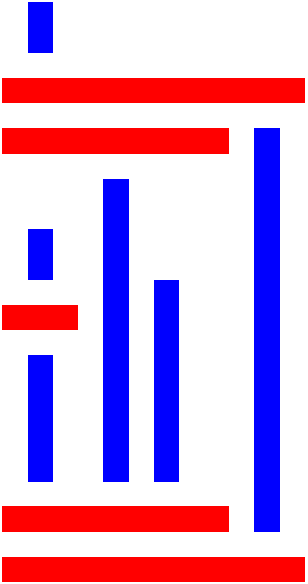


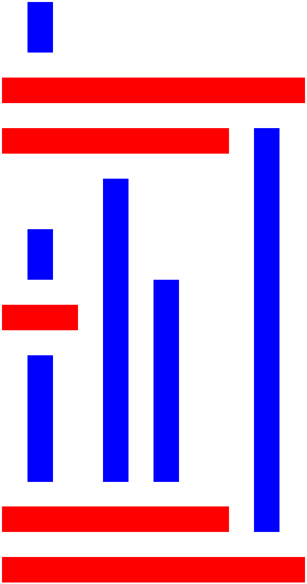








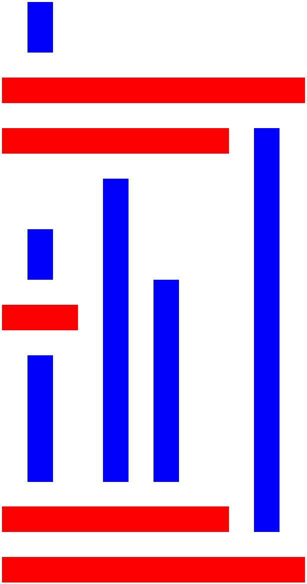




Intuition

Decidable mechanisms, $SC^{\pm}$:

- Start with partially blind counters
- Build stacks
- Add blind counters



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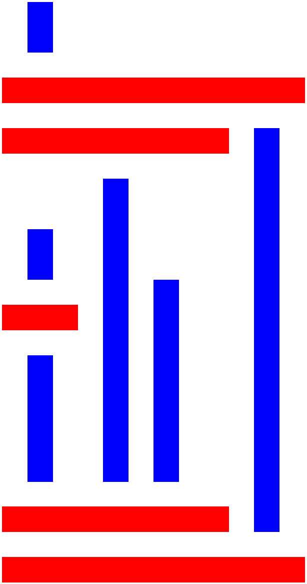

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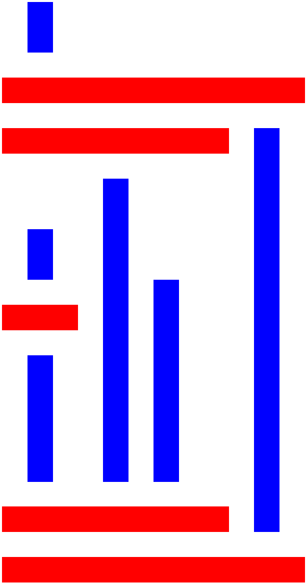

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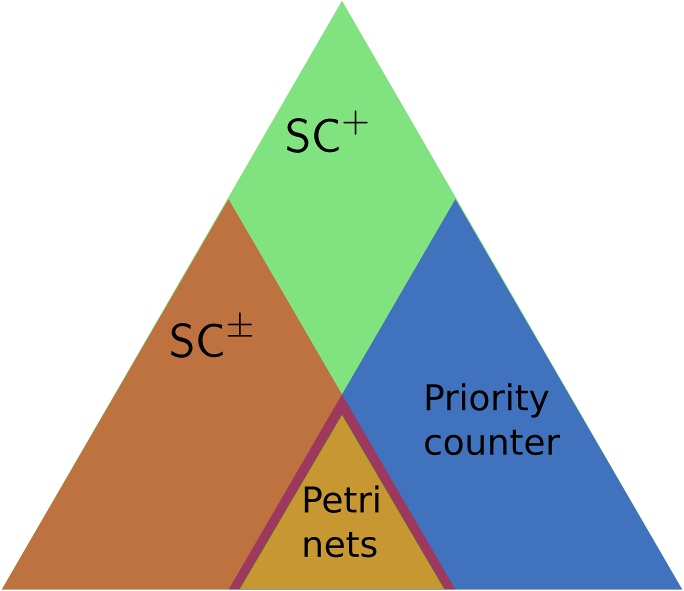
⇒ Generalize pushdown Petri nets and priority counter automata

⇒ New open problem


SC$^+$
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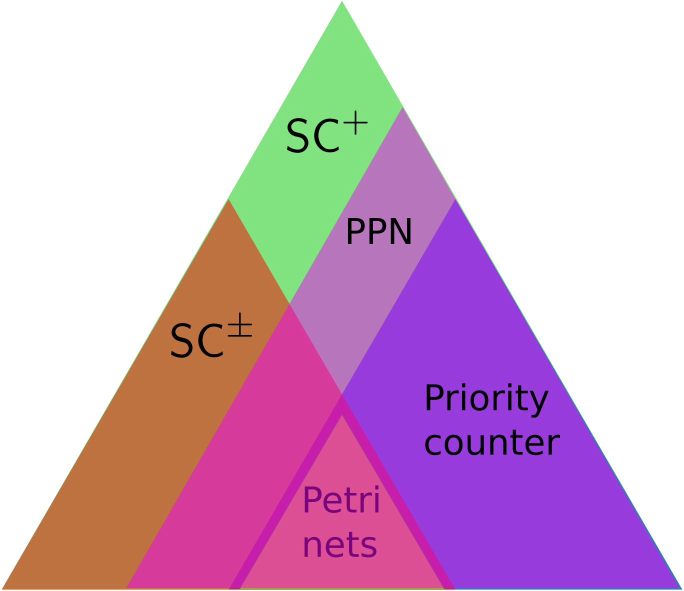
SC^{+}
PPN
SC$^{\pm}$
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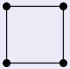
Poof: Undecidability

Theorem (Wolk 1965)

An undirected graph is a transitive forest iff it avoids as induced subgraphs:

C_4 : 

P_4 : 

$\Rightarrow$ Show Turing completeness for C_4 and P_4

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Combinatorial argument shows: equivalent to $\mathsf{SC}^{\pm}$.

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Definition of $\mathsf{SC}^{\pm}$

Smallest class with

- $\mathbb{B}^n \in \mathsf{SC}^{\pm}$
- if $M \in \mathsf{SC}^{\pm}$, then $\mathbb{B} * M, \quad \mathbb{Z} \times M \in \mathsf{SC}^{\pm}$

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Reduction

$\Psi(\mathsf{VA}(M)) \subseteq \mathsf{Prio}$ for every $M \in \mathsf{SC}^{\pm}$.

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Theorem (Reinhardt)

Reachability is decidable for priority counter machines.

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- If $\Psi(\mathsf{VA}(M)) \subseteq \mathsf{Prio}$, then $\Psi(\mathsf{VA}(M \times \mathbb{Z})) \subseteq \Psi(\mathsf{Prio})$.
- What about $\mathsf{VA}(\mathbb{B} * M)$?

Expressiveness

Algebraic extensions

Let $\mathcal{C}$ be a language class. A *$\mathcal{C}$-grammar* G consists of

- Nonterminals N, terminals T, start symbol $S \in N$
- Productions $A \to L$ with $L \subseteq (N \cup T)^*$, $L \in \mathcal{C}$

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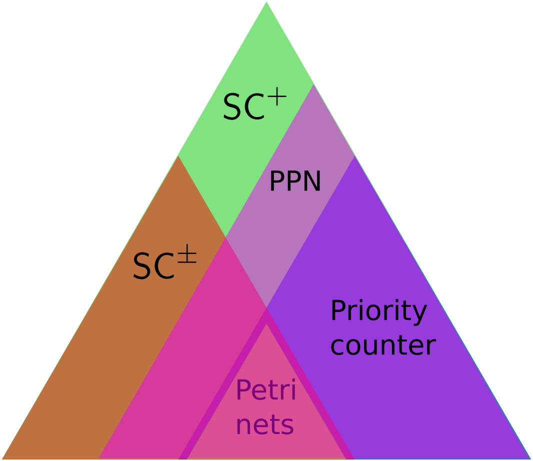
Theorem (van Leeuwen 1974)

If $\mathcal{C}$ is closed under rational transductions and Kleene star, then $\Psi(\mathsf{Alg}(\mathcal{C})) \subseteq \Psi(\mathcal{C})$.



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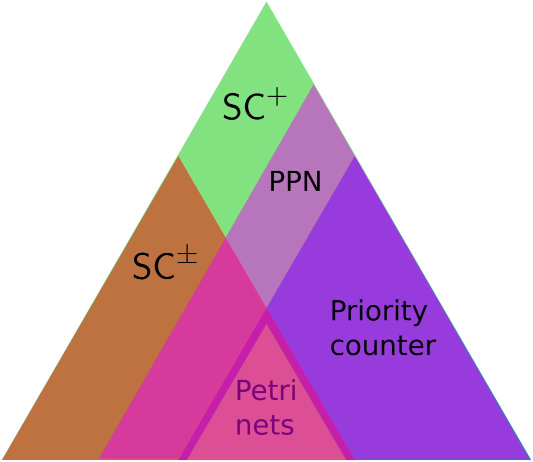

Theorem (Lohrey and Steinberg 2008)

Let Γ be a graph in which every vertex is looped. Then emptiness is decidable for $\mathbb{M}\Gamma$ if and only if Γ, minus loops, is a transitive forest.

Abstractions: Semilinear Parikh images

Semilinear Parikh images

- Numerous applications.
- Parikh's Theorem: Pushdown automata
- Ibarra + Greibach: Blind counter automata

Question

For which monoids M are all languages in $\mathsf{VA}(M)$ semilinear?

Characterization

Theorem (Buckheister, Z. 2013)

Let Γ be a graph. The following conditions are equivalent:

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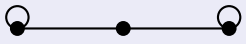
- Γ *satisfies:*
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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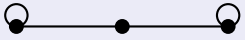
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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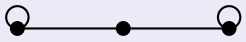
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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Characterization

Theorem (Buckheister, Z. 2013)

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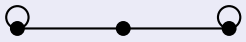
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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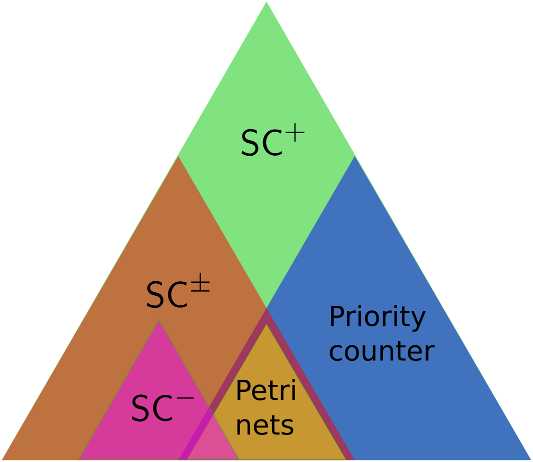
SC^{-}

Building stacks, adding blind counters


SC+
SC±
Priority counter
SC−
Petri nets

Question

For which monoids M is $\mathsf{VA}(M)$ closed under Boolean operations?

Motivation: Automatic structures

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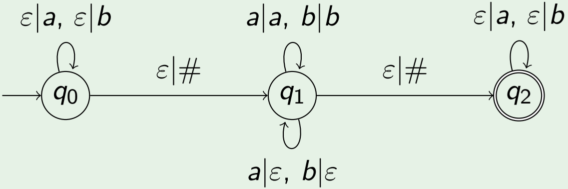
  $\Rightarrow$ new decidable structures?

Example (Transducer)



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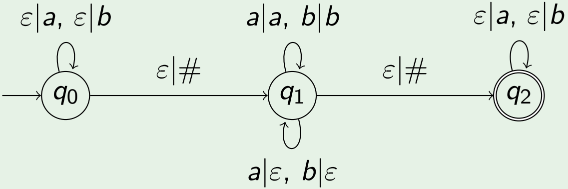

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Definition

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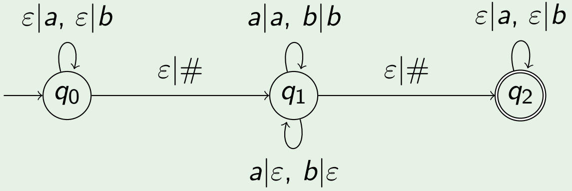

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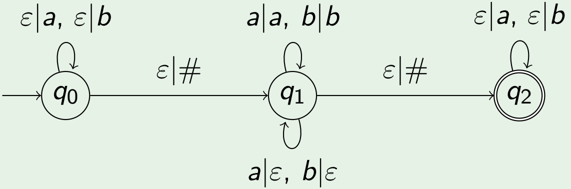

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Fact

Each VA(M) is a full trio.

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Theorem (Lohrey, Z. 2014)

If L is non-regular, then the smallest Boolean closed full trio containing L equals $\mathsf{AH}(L)$.

How to construct AH(L)

- Difficulty: Construct language of counter instructions
- Sequences over $\{+, -, 0\}$ that correspond to valid counter operations
- Only information about L: It is not regular

Idea

- Use Myhill-Nerode classes—infinitely many
- Encode counter values by Myhill-Nerode classes

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For which storage mechanisms can we avoid silent transitions?

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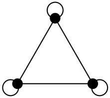
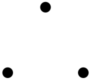


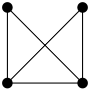
$\mathbb{Z}^3$

Blind counter



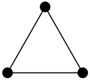
$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown



$(\mathbb{B} * \mathbb{B}) \times \mathbb{B} \times \mathbb{B}$

Pushdown + partially blind counters



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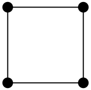

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Infinite tape (TM)

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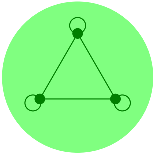

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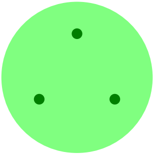

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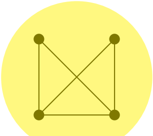

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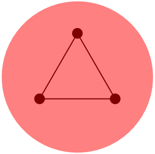

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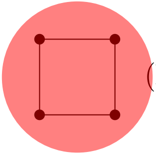

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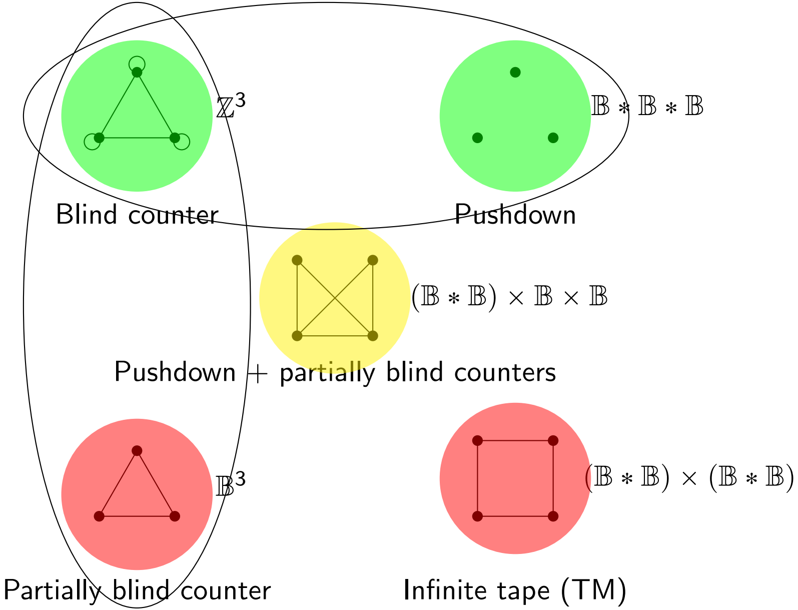

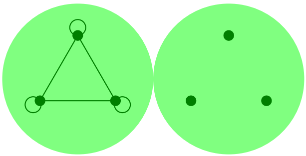

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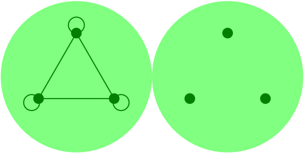

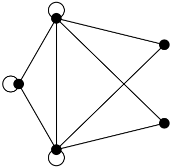

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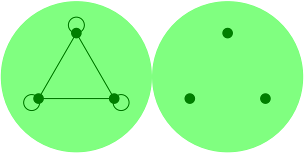

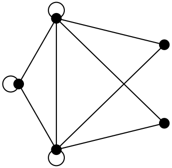

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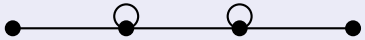
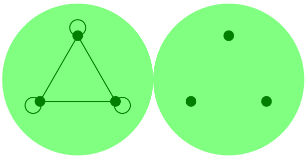
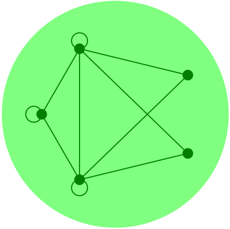
- *Γ does not contain*  *as an induced subgraph.*





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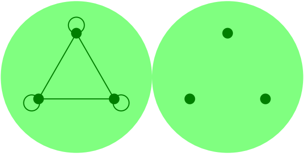

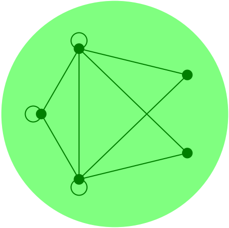

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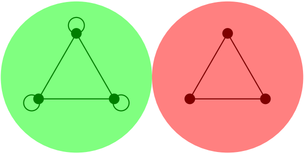

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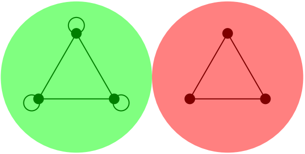

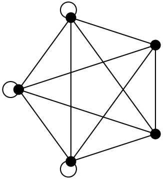

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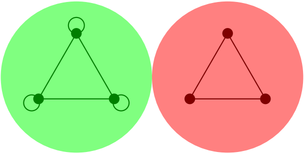

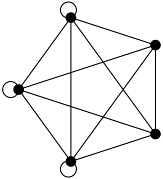

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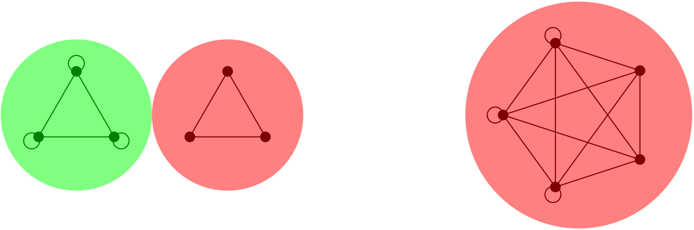

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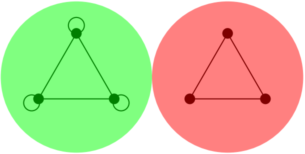

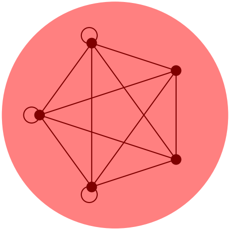

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$$\mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \quad \textit{iff} \quad r \leqslant 1.$$

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