Monoids as Storage Mechanisms

Georg Zetzsche

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INFINI Group Seminar
Example (Pushdown automaton)

- Initial state: $q_0$
- Transitions:
  - $a, \varepsilon, A \rightarrow q_1$
  - $b, \varepsilon, B \rightarrow q_0$
  - $\varepsilon, \varepsilon, \varepsilon \rightarrow q_0, q_1$

Example (Blind counter automaton)

- Initial states: $q_0, q_1, q_2$
- Transitions:
  - $a, 1 \rightarrow q_0$
  - $b, -1 \rightarrow q_1$
  - $c, 0 \rightarrow q_2$
  - $\varepsilon, 0 \rightarrow q_0, q_1, q_2$

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Example (Pushdown automaton)

\[ L = \{ w w^\text{rev} \mid w \in \{a, b\}^* \} \]
Example (Pushdown automaton)

\[
\begin{align*}
q_0 & \xrightarrow{a, \varepsilon, A} q_0 & \quad & \varepsilon, \varepsilon, \varepsilon \\
q_0 & \xrightarrow{b, \varepsilon, B} q_1 & \quad & \varepsilon, \varepsilon, \varepsilon \\
q_1 & \xrightarrow{a, A, \varepsilon} q_1 & \quad & \varepsilon, \varepsilon, \varepsilon \\
q_1 & \xrightarrow{b, B, \varepsilon} q_2 & \quad & \varepsilon, \varepsilon, \varepsilon \\
q_2 & \xrightarrow{a, \varepsilon, A} q_0 & \quad & \varepsilon, \varepsilon, \varepsilon \\
q_2 & \xrightarrow{b, \varepsilon, B} q_1 & \quad & \varepsilon, \varepsilon, \varepsilon \\
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\end{align*}
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\[L = \{ww^{\text{rev}} \mid w \in \{a, b\}^*\}\]

Example (Blind counter automaton)

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\begin{align*}
q_0 & \xrightarrow{a, 1, 0} q_0 & \quad & \varepsilon, 0, 0 \\
q_0 & \xrightarrow{b, -1, -1} q_1 & \quad & \varepsilon, 0, 0 \\
q_1 & \xrightarrow{c, 0, 1} q_2 & \quad & \varepsilon, 0, 0 \\
q_1 & \xrightarrow{\varepsilon, 0, 0} q_1 & \quad & \varepsilon, 0, 0 \\
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\end{align*}
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Example (Pushdown automaton)

\[ L = \{ ww^{rev} \mid w \in \{a, b\}^* \} \]

Example (Blind counter automaton)

\[ L = \{ a^n b^n c^n \mid n \geq 0 \} \]
Example (Partially blind counter automaton)

\[ q_0 \xrightarrow{a,1} q_0 \xrightarrow{\varepsilon,0} q_0 \xrightarrow{b,-1} q_0 \]

\[ q_0 \xrightarrow{\varepsilon,-1} q_1 \xrightarrow{\varepsilon,0} q_1 \xrightarrow{\varepsilon,0} q_1 \xrightarrow{\varepsilon,-1} q_1 \]
Example (Partially blind counter automaton)

\[ L = \{ w \in \{a, b\}^* \mid |p|_a \geq |p|_b \text{ for each prefix } p \text{ of } w \} \]
Storage mechanisms

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

Goal: General insights

Structure of storage $\Leftrightarrow$ computational properties
Storage mechanisms

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
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- Turing machines

Goal: General insights

Structure of storage ↔ computational properties

Framework

Abstract model with storage as parameter
Valence automata

**Definition**

A *monoid* is a set $M$ with

- an associative binary operation $\cdot : M \times M \rightarrow M$ and
- a neutral element $1 \in M$ ($a1 = 1a = a$ for any $a \in M$).
Valence automata

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Common generalization: Valence Automata

Valence automaton over $M$:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$. 

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- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.
- Run $q_0 \xrightarrow{w_1|m_1} q_1 \xrightarrow{w_2|m_2} \cdots \xrightarrow{w_n|m_n} q_n$ is *accepting* for $w_1 \cdots w_n$ if
  - $q_0$ is the initial state,
  - $q_n$ is a final state, and
Valence automata

Definition

A **monoid** is a set \( M \) with
- an associative binary operation \( \cdot : M \times M \to M \) and
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  - \( q_0 \) is the initial state,
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  - \( m_1 \cdots m_n = 1 \).
Valence automata

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Language class

$\text{VA}(M)$ languages accepted by valence automata over $M$. 
Classical results can now be generalized:

**Questions**

- For which storage mechanisms can we **decide emptiness**?
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- How is the complexity of decision problems affected?
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**Questions**

- For which storage mechanisms can we **decide emptiness**?
- For which do we have a particular **closure property**?
- How is the **complexity** of decision problems affected?
- For which can we compute **abstractions**?
Monoids defined by graphs

By graphs, we mean undirected graphs with loops allowed.
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Let $\Gamma = (V, E)$ be a graph. Let

$$X_\Gamma = \{a_v, \bar{a}_v \mid v \in V\}$$
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$$\cup \{xy = yx \mid x \in \{a_u, \tilde{a}_u\}, y \in \{a_v, \tilde{a}_v\}, \{u, v\} \in E\}$$
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$$M_\Gamma = X_\Gamma^*/R_\Gamma$$
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$$\mathbb{M}_\Gamma = X_\Gamma^*/R_\Gamma$$

Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^*/\{a\bar{a} = \varepsilon\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}_\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute
Examples
Examples

$\mathbb{Z}^3$
Examples

Blind counter

$\mathbb{Z}^3$
Examples

Blind counter

\[ \mathbb{Z}^3 \]
Examples

Blind counter

\[ \mathbb{Z}^3 \]

\[ \mathbb{B} \ast \mathbb{B} \ast \mathbb{B} \]
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ B \ast B \ast B \]

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Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$B \ast B \ast B$
Examples

- Blind counter
- Pushdown

- Infinite tape (TM)
- Pushdown + partially blind counters

\[ \mathbb{Z}^3 \]

\[ B \hat{\times} B \hat{\times} B \]
Examples

- Blind counter
  - \( \mathbb{Z}^3 \)

- Pushdown
  - \( B \times B \times B \)

- Partially blind counter
  - \( B^3 \)
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ \mathbb{B} \ast \mathbb{B} \ast \mathbb{B} \]

Partially blind counter

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Examples

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Pushdown

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Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ B \times B \times B \]

Partially blind counter

\[ B^3 \]

\[(B \times B) \times (B \times B)\]
Examples

Blind counter

Pushdown

Partially blind counter

Infinite tape (TM)
Examples

- **Blind counter**: $\mathbb{Z}^3$
- **Pushdown**: $B \times B \times B$
- **Partially blind counter**: $B^3$
- **Infinite tape (TM)**: $(B \times B) \times (B \times B)$
Examples

Blind counter

Pushdown

Partially blind counter

Infinite tape (TM)
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Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ B \times B \times B \]

Pushdown + partially blind counters

\[ (B \times B) \times B \times B \]

Partially blind counter

\[ B^3 \]

Infinite tape (TM)

\[ (B \times B) \times (B \times B) \]
The emptiness problem

Given a valence automaton over $M$, does it accept any word?
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Important problem

- Type of reachability problem
- Necessary for many other decision problems.
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For which storage mechanisms is the emptiness problem decidable?
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Obstacle

Pushdown + partially blind counters
The emptiness problem

Given a valence automaton over \( M \), does it accept any word?

Important problem

- Type of reachability problem
- Necessary for many other decision problems.

Question

For which storage mechanisms is the emptiness problem decidable?

Obstacle

Pushdown + partially blind counters

Decidability a long-standing open problem
Simplest graphs for pushdown + counters

One can show: These can simulate pushdown + one counter

Theorem (Z. 2015)

Let $\Gamma$ be PPN-free. Then the following are equivalent:

- Emptiness is decidable for valence automata over $M_\Gamma$.
- $\Gamma$, minus loops, is a transitive forest.
Simplest graphs for pushdown + counters

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- We call these *PPN-graphs* (for “pushdown Petri net”).
Simplest graphs for pushdown + counters

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- Without them as induced subgraphs: *PPN-free*.
Simplest graphs for pushdown + counters

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- We call these *PPN-graphs* (for “pushdown Petri net”).
- Without them as induced subgraphs: *PPN-free*.

**Theorem (Z. 2015)**

Let $\Gamma$ be *PPN-free*. Then the following are equivalent:

- *Emptiness is decidable for valence automata over $\mathbb{M}\Gamma$.*
- $\Gamma$, minus loops, is a *transitive forest*. 
Decidable mechanisms, SC ˘:

Start with partially blind counters
Build stacks
Add blind counters

Reduction to priority counter automata of Reinhardt

Left open, SC `:

Start with partially blind counters
Build stacks
Add partially blind counters

Generalize pushdown Petri nets and priority counter automata

New open problem
Intuition

Decidable mechanisms, SC

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Intuition
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Intuition

Decidable mechanisms, $\text{SC}^\pm$:
- Start with partially blind counters
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- Add blind counters
Decidable mechanisms, SC$^\pm$:
- Start with partially blind counters
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$\Rightarrow$ Reduction to priority counter automata of Reinhardt
Intuition

Decidable mechanisms, $SC^\pm$:

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$\Rightarrow$ Reduction to priority counter automata of Reinhardt

Left open, $SC^+$:

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Intuition

Decidable mechanisms, $SC^\pm$:
- Start with partially blind counters
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  $\Rightarrow$ Reduction to priority counter automata of Reinhardt

Left open, $SC^+$:
- Start with partially blind counters
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- Add partially blind counters
  $\Rightarrow$ Generalize pushdown Petri nets and priority counter automata
  $\Rightarrow$ New open problem
Petri nets
Priority counter

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**Theorem (Wolk 1965)**

An undirected graph is a transitive forest iff it avoids as induced subgraphs:

- $C_4$:
- $P_4$:

$\Rightarrow$ Show Turing completeness for $C_4$ and $P_4$
Poof: Decidability

Decidability

Combinatorial argument shows: equivalent to $\text{SC}^\pm$. 
Decidability

Combinatorial argument shows: equivalent to $\text{SC}^\pm$.

Definition of $\text{SC}^\pm$

Smallest class with

- $\mathbb{B}^n \in \text{SC}^\pm$
- if $M \in \text{SC}^\pm$, then $\mathbb{B} \ast M$, $\mathbb{Z} \times M \in \text{SC}^\pm$
Poof: Decidability

Decidability

Combinatorial argument shows: equivalent to $SC^\pm$.

Definition of $SC^\pm$

Smallest class with

- $B^n \in SC^\pm$
- if $M \in SC^\pm$, then $B \star M$, $\mathbb{Z} \times M \in SC^\pm$

Reduction

$\Psi(\text{VA}(M)) \subseteq \text{Prio}$ for every $M \in SC^\pm$. 
Priority counter machines

- Automaton with \( n \) counters

Language class: Prio

Theorem (Reinhardt)
Reachability is decidable for priority counter machines.
Priority counter machines

- Automaton with \( n \) counters
- Counters stay \( \geq 0 \)
Priority counter machines

- Automaton with $n$ counters
- Counters stay $\geq 0$
- Instructions:
  - $\text{inc}_i$: increment counter $i$
  - $\text{dec}_i$: decrement counter $i$
  - $\text{zero}_i$: test all the counters $1, \ldots, i$ for zero
Priority counter machines

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- Language class: Prio
Priority counter machines

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Smallest class with
- $\mathbb{B}^n \in \text{SC}^\pm$
- if $M \in \text{SC}^\pm$, then $\mathbb{B} \star M, \mathbb{Z} \times M \in \text{SC}^\pm$

Observations
- $\text{VA}(\mathbb{B}^n) \subseteq \text{Prio}$, hence $\psi(\text{VA}(\mathbb{B}^n)) \subseteq \psi(\text{Prio})$. 
**Definition of SC\(^\pm\)**

Smallest class with

- \(B^n \in SC^\pm\)
- if \(M \in SC^\pm\), then \(B \ast M, \quad \mathbb{Z} \times M \in SC^\pm\)

**Observations**

- \(VA(B^n) \subseteq Prio\), hence \(\Psi(VA(B^n)) \subseteq \Psi(Prio)\).
- If \(\Psi(VA(M)) \subseteq Prio\), then \(\Psi(VA(M \times \mathbb{Z})) \subseteq \Psi(Prio)\).
### Definition of SC$^\pm$

Smallest class with
- $\mathbb{B}^n \in \text{SC}^\pm$
- if $M \in \text{SC}^\pm$, then $\mathbb{B} \ast M$, $\mathbb{Z} \times M \in \text{SC}^\pm$

### Observations

- $\text{VA}(\mathbb{B}^n) \subseteq \text{Prio}$, hence $\Psi(\text{VA}(\mathbb{B}^n)) \subseteq \Psi(\text{Prio})$.
- If $\Psi(\text{VA}(M)) \subseteq \text{Prio}$, then $\Psi(\text{VA}(M \times \mathbb{Z})) \subseteq \Psi(\text{Prio})$.
- What about $\text{VA}(\mathbb{B} \ast M)$?
Expressiveness

Algebraic extensions

Let $\mathcal{C}$ be a language class. A $\mathcal{C}$-grammar $G$ consists of

- Nonterminals $N$, terminals $T$, start symbol $S \in N$
- Productions $A \rightarrow L$ with $L \subseteq (N \cup T)^*$, $L \in \mathcal{C}$
Expressiveness

Algebraic extensions

Let $C$ be a language class. A $C$-grammar $G$ consists of

- Nonterminals $N$, terminals $T$, start symbol $S \in N$
- Productions $A \rightarrow L$ with $L \subseteq (N \cup T)^*$, $L \in C$

$$uAv \Rightarrow uwv \quad \text{whenever } w \in L.$$
Expressiveness

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- Nonterminals $N$, terminals $T$, start symbol $S \in N$
- Productions $A \rightarrow L$ with $L \subseteq (N \cup T)^*$, $L \in C$
  
  $uAv \Rightarrow uwv$ whenever $w \in L$.
- Generated language: $\{w \in T^* \mid S \Rightarrow^* w\}$.

Such languages are algebraic over $C$, class denoted $\text{Alg}_p C$.

Theorem (Z. 2015)

Theorem (van Leeuwen 1974)

If $C$ is closed under rational transductions and Kleene star, then

$\Psi_p \text{Alg}_p C \subseteq \Psi_p C$. 

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Expressiveness

Algebraic extensions

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Theorem (Z. 2015)

$$\text{VA}(\mathbb{B} \ast \mathbb{B} \ast M) = \text{Alg}(\text{VA}(M)).$$
Expressiveness

Algebraic extensions

Let $C$ be a language class. A $C$-grammar $G$ consists of

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Theorem (Z. 2015)

$$\text{VA}(\mathbb{B} \ast \mathbb{B} \ast M) = \text{Alg}(\text{VA}(M)).$$

Theorem (van Leeuwen 1974)

*If $C$ is closed under rational transductions and Kleene star, then
\[
\Psi(\text{Alg}(C)) \subseteq \Psi(C).
\]
Theorem (Lohrey and Steinberg 2008)

Let $\Gamma$ be a graph in which every vertex is looped. Then emptiness is decidable for $M_{\Gamma}$ if and only if $\Gamma$, minus loops, is a transitive forest.
Theorem (Lohrey and Steinberg 2008)

Let $\Gamma$ be a graph in which every vertex is looped. Then emptiness is decidable for $\mathbb{M}\Gamma$ if and only if $\Gamma$, minus loops, is a transitive forest.
### Semilinear Parikh images

- Numerous applications.
- Parikh’s Theorem: Pushdown automata
- Ibarra + Greibach: Blind counter automata

### Question

For which monoids $M$ are all languages in $\text{VA}(M)$ semilinear?
Characterization

Theorem (Buckheister, Z. 2013)

Let $\Gamma$ be a graph. The following conditions are equivalent:

- All languages in $\mathcal{VA}(\mathbb{M}\Gamma)$ are semilinear.
- $\Gamma$ satisfies:
  1. $\Gamma$ contains neither $\bullet\bullet$ nor $\bullet\bullet\bullet\bullet\bullet$ as an induced subgraph and
  2. $\Gamma$, minus loops, is a transitive forest.
Characterization

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- $\Gamma$ satisfies:
  1. $\Gamma$ contains neither $\bullet\bullet$ nor $\circ\circ\circ$ as an induced subgraph and
  2. $\Gamma$, minus loops, is a transitive forest.
- $\text{VA}(B \times B) \not\subseteq \text{VA}(M\Gamma)$
Characterization

Theorem (Buckheister, Z. 2013)

Let \( \Gamma \) be a graph. The following conditions are equivalent:

- All languages in \( \text{VA}(\mathbb{M}\Gamma) \) are semilinear.
- \( \Gamma \) satisfies:
  1. \( \Gamma \) contains neither \( \bullet \bullet \) nor \( \bullet \bullet \bullet \bullet \) as an induced subgraph and
  2. \( \Gamma \), minus loops, is a transitive forest.
- \( \text{VA}(\mathbb{B} \times \mathbb{B}) \nsubseteq \text{VA}(\mathbb{M}\Gamma) \)
- \( \text{VA}(\mathbb{M}\Gamma) \subseteq \text{VA}(\mathbb{M}) \) for some \( \mathbb{M} \in \text{SC}^- \).
Characterization

Theorem (Buckheister, Z. 2013)

Let $\Gamma$ be a graph. The following conditions are equivalent:

- All languages in $\mathrm{VA}(\mathcal{M}\Gamma)$ are semilinear.
- $\Gamma$ satisfies:
  1. contains neither $\bullet\bullet$ nor $\bullet\circ\bullet\circ$ as an induced subgraph and
  2. minus loops, is a transitive forest.

- $\mathrm{VA}(\mathcal{B} \times \mathcal{B}) \not\subseteq \mathrm{VA}(\mathcal{M}\Gamma)$
- $\mathrm{VA}(\mathcal{M}\Gamma) \subseteq \mathrm{VA}(M)$ for some $M \in \mathcal{SC}^-$. 

$\mathcal{SC}^-$

Building stacks, adding blind counters
Petri nets
Priority counter

SC^+
SC^±
SC^-

Petri nets
Priority counter
**Question**

For which monoids $M$ is $\text{VA}(M)$ closed under Boolean operations?

**Motivation: Automatic structures**

- Infinite structures described by finite automata
- Decidable first-order logic
Question
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Motivation: Automatic structures

- Infinite structures described by finite automata
- Decidable first-order logic
- If $\text{VA}(M)$ is Boolean closed and has decidable emptiness:
  - valence automata over $M$ instead of finite automata
  - new decidable structures?
Example (Transducer)

\[
\begin{align*}
\varepsilon | a, \varepsilon | b & \quad a | a, b | b & \quad \varepsilon | a, \varepsilon | b \\
q_0 & \quad \varepsilon | \# & \quad \varepsilon | \# & \quad q_2 \\
& \quad a | \varepsilon, b | \varepsilon
\end{align*}
\]
Definition

- **Rational transduction**: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq X^*$, let

$$ TL = \{ y \in Y^* \mid \exists x \in L : (x, y) \in T \} $$
Example (Transducer)

\[ T(A) = \{(x, u\#v\#w) \mid u, v, w, x \in \{a, b\}^*, \ v \leq x\} \]

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- \( C \) is a full trio if \( LR \in C \) for each \( L \in C \) and rational transduction \( R \).
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Fact
Each VA(M) is a full trio.

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RE(C): Accepted by Turing machine with oracle $L \in C$.

**Definition**

Arithmetical hierarchy:

$$\Sigma_1 = \text{RE}, \quad \Sigma_{n+1} = \text{RE}(\Sigma_n) \text{ for } n \geq 0, \quad \text{AH} = \bigcup_{n \geq 0} \Sigma_n.$$
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Relative arithmetical hierarchy:

$$\Sigma_1(L) = \text{RE}(L), \quad \Sigma_{n+1}(L) = \text{RE}(\Sigma_n(L)) \text{ for } n \geq 0, \quad \text{AH}(L) = \bigcup_{n \geq 0} \Sigma_n(L).$$
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**Definition**

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Relative arithmetical hierarchy:

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**Theorem (Lohrey, Z. 2014)**

*If $L$ is non-regular, then the smallest Boolean closed full trio containing $L$ equals $\text{AH}(L)$.*
## How to construct $\text{AH}(L)$

- **Difficulty**: Construct language of counter instructions
- **Sequences** over $\{+,-,0\}$ that correspond to valid counter operations
- **Only information** about $L$: It is not regular

## Idea

- **Use** Myhill-Nerode classes—infinitely many
- **Encode** counter values by Myhill-Nerode classes
Silent transitions

A transition that reads no input is called *silent transition* or \( \varepsilon \)-transition.
Silent transitions

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Important problem

- Can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.
Silent transitions

A transition that reads no input is called *silent transition* or *ε-transition*.

**Important problem**

- Can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.

**Question**

For which storage mechanisms can we avoid silent transitions?
Examples, again

\[
\begin{align*}
Z^3 & \quad B \cdot B \cdot B \\
\text{Blind counter} & \quad \text{Pushdown} \\
(B \cdot B) \times B & \times B \\
\text{Pushdown + partially blind counters} & \quad \text{Partially blind counter} \\
B^3 & \quad (B \cdot B) \times (B \cdot B) \\
\text{Infinite tape (TM)} & \quad \text{Infinite tape (TM)}
\end{align*}
\]
Examples, again

- **Blind counter**: $\mathbb{Z}^3$
- **Pushdown**: $B \ast B \ast B$
- **Pushdown + partially blind counters**: $(B \ast B) \times B \times B$
- **Partially blind counter**: $B^3$
- **Infinite tape (TM)**: $(B \ast B) \times (B \ast B)$
Examples, again

- $\mathbb{Z}^3$
- $\mathbb{B} \times \mathbb{B} \times \mathbb{B}$
- $\mathbb{B}^3$
- $(\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \times \mathbb{B}$
- $(\mathbb{B} \times \mathbb{B}) \times (\mathbb{B} \times \mathbb{B})$

- Blind counter
- Pushdown
- Pushdown + partially blind counters
- Partially blind counter
- Infinite tape (TM)
Theorem (Z. 2013)

Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
- no two unlooped vertices are adjacent.

The following conditions are equivalent:

- Silent transitions can be avoided over $M_\Gamma$.
- $\Gamma$ does not contain $P_{SC}$ as an induced subgraph.
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Then the following conditions are equivalent:
- Silent transitions can be avoided over $M\Gamma$.
- $\Gamma$ does not contain $\bullet \circlearrowright \bullet$ as an induced subgraph.
- $M\Gamma \in SC^-$. 
Theorem (Z. 2013)

Let \( \Gamma \) be a graph such that between any two distinct vertices, there is an edge.

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Let $\Gamma$ be a graph such that between any two distinct vertices, there is an edge. Then $\text{VA}(M\Gamma) = \text{VA}^+(M\Gamma)$ if and only if the number of unlooped nodes is $\leq 1$. 

Georg Zetzsche (LSV Cachan)
Theorem (Z. 2013)

Let \( \Gamma \) be a graph such that between any two distinct vertices, there is an edge. Then \( VA(\mathbb{M}\Gamma) = VA^+(\mathbb{M}\Gamma) \) if and only if the number of unlooped nodes is \( \leq 1 \).
Theorem (Z. 2013)

Let $\Gamma$ be a graph such that between any two distinct vertices, there is an edge. Then $VA(M\Gamma) = VA^+(M\Gamma)$ if and only if the number of unlooped nodes is $\leq 1$. In other words:

$$VA(B^r \times \mathbb{Z}^s) = VA^+(B^r \times \mathbb{Z}^s) \text{ iff } r \leq 1.$$
**Conclusion**

**Valence automata**
- Generalize various automata with storage
- Meaningful characterizations of computational properties
- Reveal natural models with interesting properties
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Ongoing work
- For which storage mechanisms is FO+Reach decidable?
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Ongoing work

- For which storage mechanisms is FO+Reach decidable?

Thank you for your attention!