On Boolean closed full trios and rational Kripke frames

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Closure properties

Common closure properties

- Homomorphism: $h : \Sigma^* \to \Gamma^*$, replaces letters by words
- Inverse homomorphism: $\{w \in \Sigma^* \mid h(w) \in L\}$
- Intersection with regular sets.
- Boolean operations: union, intersection, complementation.

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Examples

REG, CF, LIN, Petri net languages, blind multicounter languages, classes of various grammar types, etc.

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- Automatic structures beyond regular languages
- Complementation closure for union closed full trios

$\mathrm{RE}(\mathcal{C})$: Accepted by Turing machine with oracle $L \in \mathcal{C}$.

Definition

Arithmetical hierarchy:

$$\Sigma_0 = \mathrm{REC}, \qquad \Sigma_{n+1} = \mathrm{RE}(\Sigma_n) \text{ for } n \geqslant 0, \qquad \mathrm{AH} = \bigcup_{n \geqslant 0} \Sigma_n.$$

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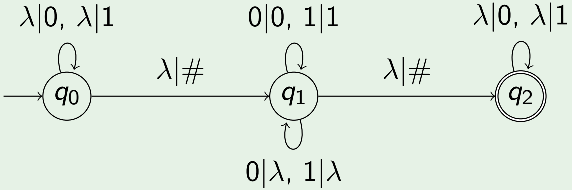
Theorem

Let $\mathcal{T}$ be a Boolean closed full trio. If $\mathcal{T}$ contains any non-regular language L, then $\mathcal{T}$ includes $\mathsf{AH}(L)$.

Example (Transducer)



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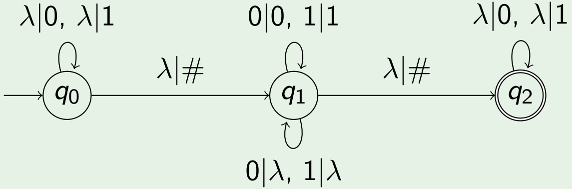

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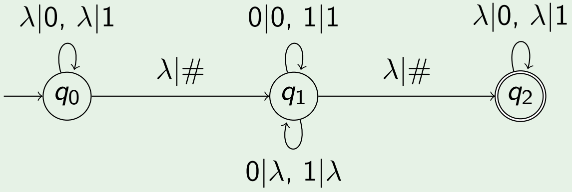

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{y \in X^* \mid \exists x \in L : (x, y) \in T\}$$

Theorem (Nivat 1968)

A language class is a full trio iff it is closed under rational transductions.

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Let $\Delta = \{+, -, z\}$: increment, decrement, and zero test.

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Theorem (Myhill-Nerode)

L is regular if and only if $\equiv_L$ has finite index.

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Idea: In order to obtain C, construct $\hat{C}_L$:

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Let $\hat{C}_L$ be the set of all words

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with $\delta_i \in \Delta$, $v_i \in X^*$, $u_j \in X^*$

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Observation

If L is non-regular, C can be obtained from $\hat{C}_L$.

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$$W_1 = \{u\#v\#w \mid u, v, w \in X^*,\ uw \in L\},$$
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$$\begin{aligned} M = & \{v_1{+}v_2\#u_1\#u_2 \mid v_1\#u_1 \in P,\ v_2\#u_2 \in P\} \\ & \cup \{v_1{-}v_2\#u_1\#u_2 \mid v_1\#u_2 \in P,\ v_2\#u_1 \in P\} \\ & \cup \{v_1 z v_2\#u_1\#u_2 \mid v_1\#v_2 \in P,\ v_1\#u_1 \in P,\ u_2 \in X^*\} \end{aligned}$$

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$$E' = \{v_1 \delta v_2 \# r u_1 \# u_2 \# s \mid v_1 \delta v_2 \# u_1 \# u_2 \in M, r, s \in (X^* \#)^*\}$$

$$E = \left[(X^* \Delta X^* \# (X^* \#)^* \setminus E'\right]$$

Proof IV

Let M (*matching*) be the set of all words $v_1 \delta v_2 \# u_1 \# u_2$, $v_1, v_2, u_1, u_2 \in X^*$, with

- if $\delta = +$, then $v_1 \equiv_L u_1$ and $v_2 \equiv_L u_2$,
- if $\delta = -$, then $v_1 \equiv_L u_2$ and $v_2 \equiv_L u_1$, and
- if $\delta = z$, then $v_1 \equiv_L v_2 \equiv_L u_1$.

$$\begin{aligned} M = {} & \{v_1{+}v_2\#u_1\#u_2 \mid v_1\#u_1 \in P,\ v_2\#u_2 \in P\} \\ & \cup \{v_1{-}v_2\#u_1\#u_2 \mid v_1\#u_2 \in P,\ v_2\#u_1 \in P\} \\ & \cup \{v_1 z v_2\#u_1\#u_2 \mid v_1\#v_2 \in P,\ v_1\#u_1 \in P,\ u_2 \in X^*\} \end{aligned}$$

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Proof V

Let N (*no error*) be the set of words $v_0\delta_1 v_1 \cdots \delta_m v_m \# u_0 \# \cdots u_n \#$ such that for every $1 \leqslant i \leqslant m$, there is a $1 \leqslant j \leqslant n$ with $v_{i-1}\delta_i v_i \# u_{j-1} \# u_j \in M$ and if $\delta_i = z$, then $v_{i-1} \equiv_L u_0$.

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$$N' = \{w \in (X^*\Delta)^* v_1 \delta v_2 (\Delta X^*)^* \# u_0 \# \cdots u_n \# \mid v_1 \delta v_2 \# u_0 \# \cdots u_n \# \in E\},$$

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Now we have

$$\hat{C}_L = N \cap (X^*\Delta)^* X^* \# S.$$

Hence, $C \in \mathcal{T}$.

Proof VI

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For AH$(L) \subseteq \mathcal{T}$: show that $K \in \mathcal{T}$ implies RE$(K) \subseteq \mathcal{T}$ (as above).

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Let L be non-regular. The smallest Boolean closed full trio containing L is $\mathrm{AH}(L)$.

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Let $\mathcal{T}$ be generated by L. It consists of RL for rational transductions R. Hence, $\mathcal{T}$ is union-closed and $\mathcal{T} \subseteq \mathsf{RE}(L) \subsetneq \mathsf{AH}(L)$. If $\mathcal{T}$ were complementation closed, it would contain $\mathsf{AH}(L)$, contradiction! □

Application: Valence automata

Definition

A monoid is a set M together with an associative operation and a neutral element.

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Valence Automata

Valence automaton over M:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.
- Run $q_0 \xrightarrow{w_1|m_1} q_1 \xrightarrow{w_2|m_2} \cdots \xrightarrow{w_n|m_n} q_n$ is *accepting for* $w_1 \cdots w_n$ if
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Language class

VA(M) languages accepted by valence automata over M.

Corollary

Let M be a finitely generated monoid. The following are equivalent:

1. $\mathsf{VA}(M)$ *is complementation closed.*
2. $\mathsf{VA}(M) = \mathsf{REG}$.
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If M is finitely generated, $\mathrm{VA}(M)$ is a principal full trio. Equivalence of 2 and 3 has been shown by Render (2010) and Z. (2011). □

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Improvement

In order to construct languages over $\{0, 1\}$, three fixed rational transductions suffice.

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Theorem

Let $X = \{0, 1\}$. There is a Kripke frame with

- *X^* as its set of worlds and*
- *rational transductions $R, S, T \subseteq X^* \times X^*$ as modalities*

such that for any non-regular L, in the Kripke structure $\mathcal{K} = (X^*, R, S, T, L)$, for each $K \in \mathrm{AH}(L)$, there is a φ with $[\![\varphi]\!]_{\mathcal{K}} = K$.

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Synchronous rational transductions

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- Decidable MSO-theory yields decidability

Definition

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Example: Identity language of a finitely generated infinite group.

Theorem

Let $L \subseteq \{0, 1\}^*$ be a non-regular language with a neutral word. Using synchronous rational transductions and Boolean operations, one can construct a non-recursively enumerable language from L. If, in addition, L is recursive, one can construct a Σ_n-hard language from L.