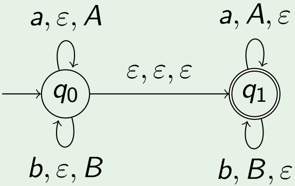
The Emptiness Problem for Valence Automata or: Another Decidable Extension of Petri Nets

Georg Zetzsche

Technische Universität Kaiserslautern

Reachability Problems 2015

Example (Pushdown automaton)



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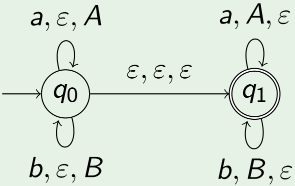

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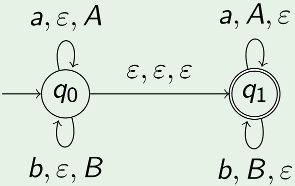

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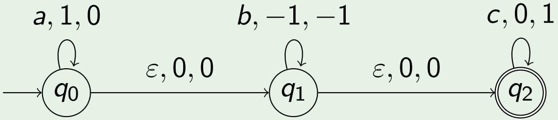

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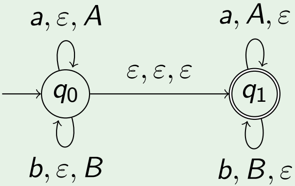

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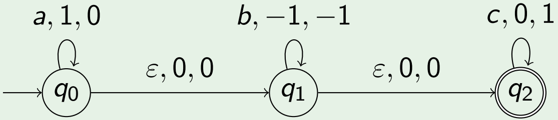

$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

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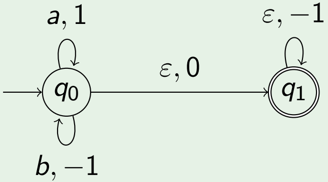

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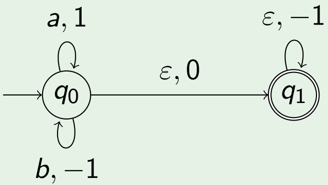

$$L = \{w \in \{a, b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for each prefix } p \text{ of } w\}$$

Storage mechanisms

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

Goal: General insights

Structure of storage $\Leftrightarrow$ computational properties

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Framework

Abstract model with storage as parameter

Valence automata

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- an associative binary operation $\cdot\colon M \times M \to M$ and
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Language class

VA(M) languages accepted by valence automata over M.

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- For which do we have a particular closure property?
- How is the complexity of decision problems affected?
- For which can we compute abstractions?

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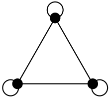
Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^* / \{a\bar{a} = \varepsilon\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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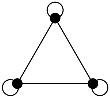

$\mathbb{Z}^3$

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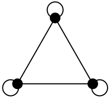

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Blind counter

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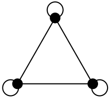

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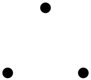

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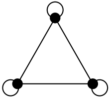

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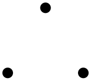

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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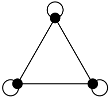

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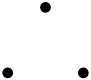

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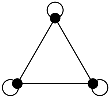

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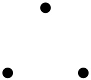

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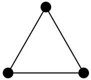

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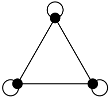

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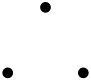

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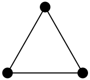

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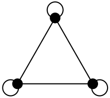

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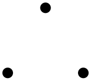

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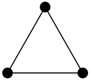

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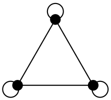

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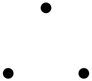

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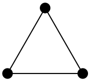

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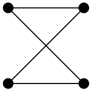

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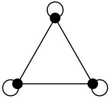

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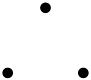

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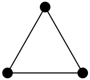

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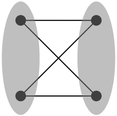

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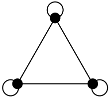

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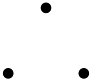

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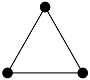

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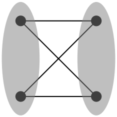

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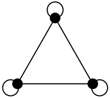

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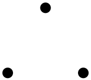

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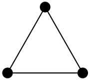

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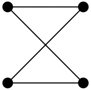

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Infinite tape (TM)

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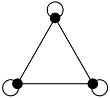

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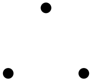

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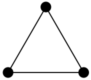

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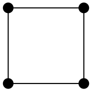

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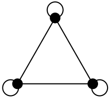

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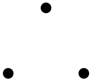

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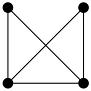

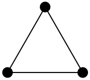

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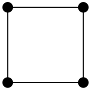
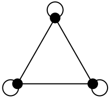


$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

Infinite tape (TM)

Examples



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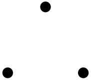
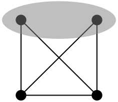
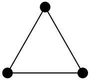


$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown





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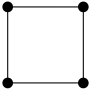

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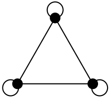

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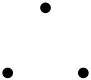

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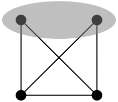

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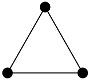

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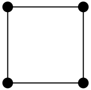

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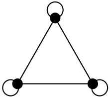

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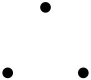

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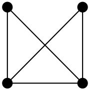

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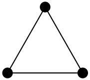

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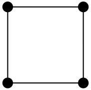

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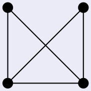

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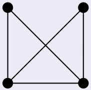
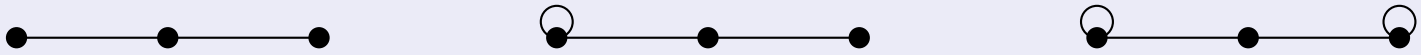


Pushdown + partially blind counters

Decidability a long-standing open problem

Simplest graphs for pushdown + counters



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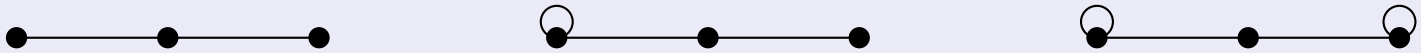

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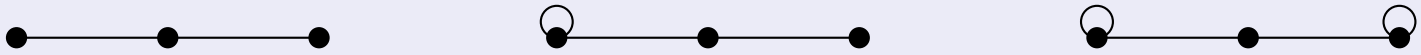

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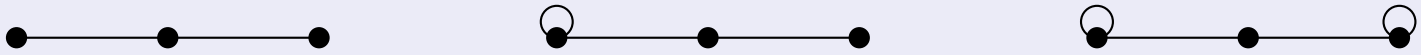

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Let Γ *be PPN-free. Then the following are equivalent:*

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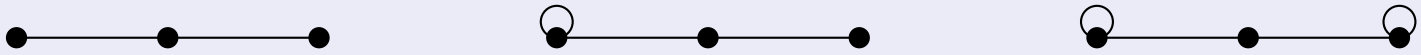

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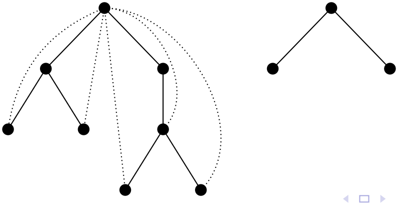
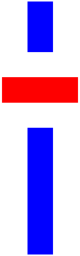
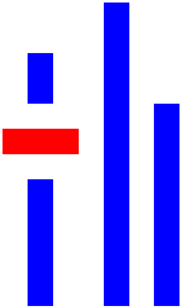
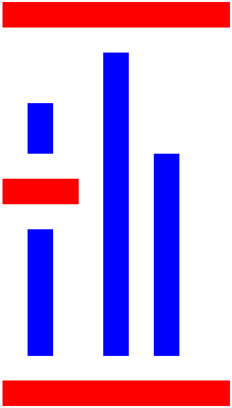
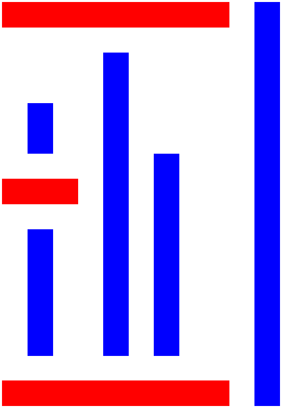
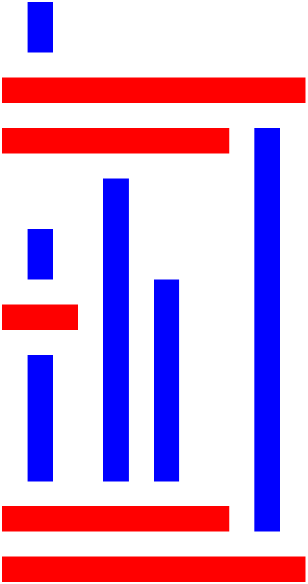


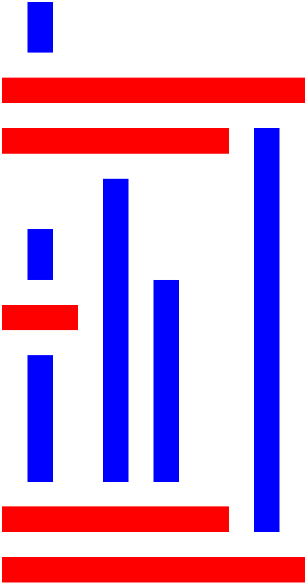








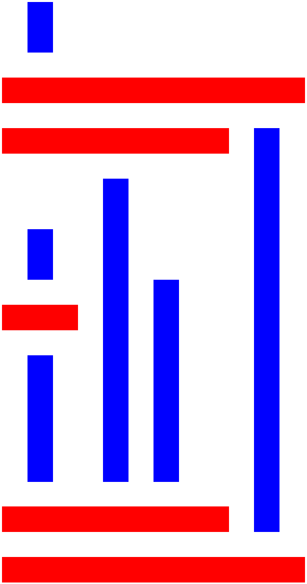




Intuition

Decidable mechanisms, $SC^{\pm}$:

- Start with partially blind counters
- Build stacks
- Add blind counters



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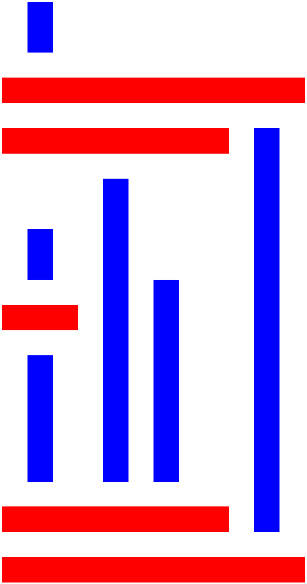

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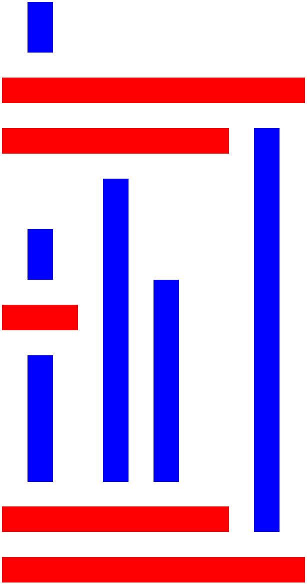

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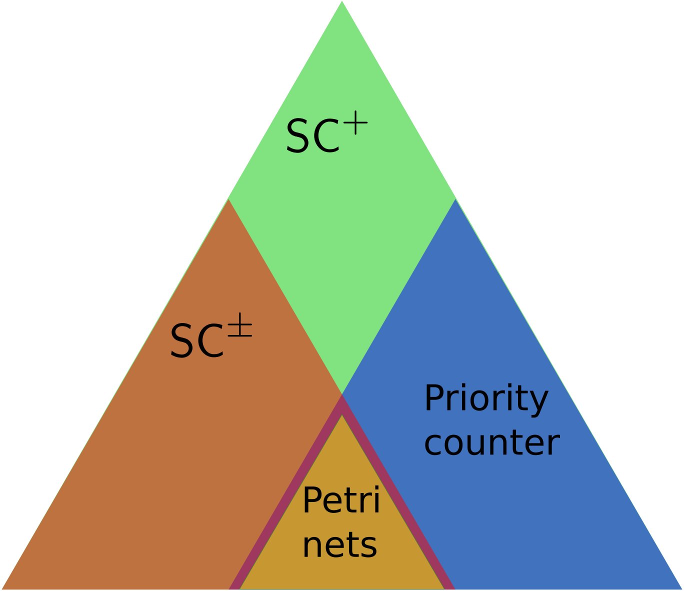
$\Rightarrow$ Generalize pushdown Petri nets and priority counter automata

$\Rightarrow$ New open problem


SC$^+$
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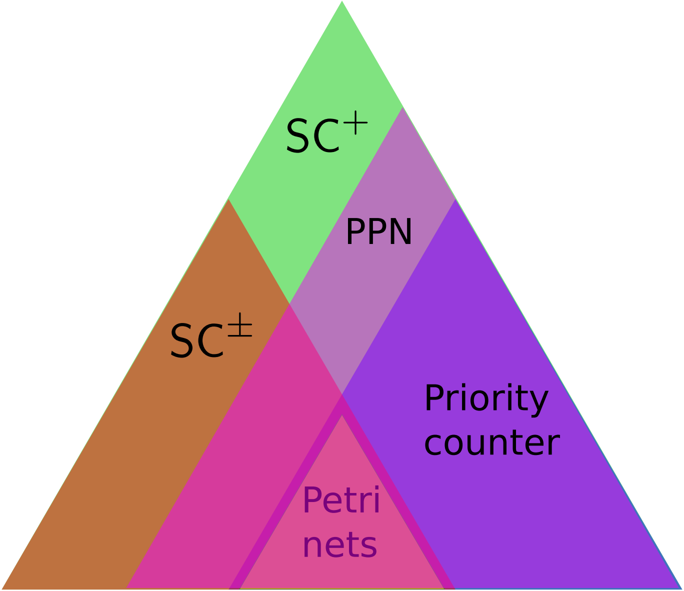
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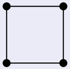
Poof: Undecidability

Theorem (Wolk 1965)

An undirected graph is a transitive forest iff it avoids as induced subgraphs:

C_4 : 

P_4 : 

$\Rightarrow$ Show Turing completeness for C_4 and P_4

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Combinatorial argument shows: equivalent to $SC^{\pm}$.

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Definition of $\mathsf{SC}^{\pm}$

Smallest class with

- $\mathbb{B}^n \in \mathsf{SC}^{\pm}$
- if $M \in \mathsf{SC}^{\pm}$, then $\mathbb{B} * M, \quad \mathbb{Z} \times M \in \mathsf{SC}^{\pm}$

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Reduction

$\Psi(\mathsf{VA}(M)) \subseteq \mathsf{Prio}$ for every $M \in \mathsf{SC}^{\pm}$.

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Theorem (Reinhardt)

Reachability is decidable for priority counter machines.

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- What about $\mathsf{VA}(\mathbb{B} * M)$?

Expressiveness

Algebraic extensions

Let $\mathcal{C}$ be a language class. A *$\mathcal{C}$-grammar* G consists of

- Nonterminals N, terminals T, start symbol $S \in N$
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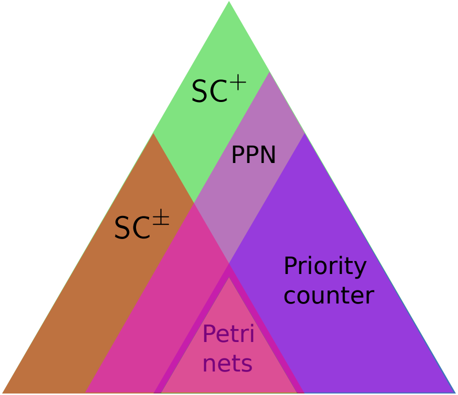
Theorem (van Leeuwen 1974)

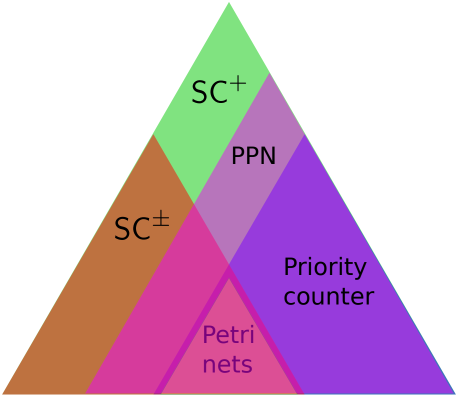
If $\mathcal{C}$ is closed under rational transductions and Kleene star, then $\Psi(\mathsf{Alg}(\mathcal{C})) \subseteq \Psi(\mathcal{C})$.

Conclusion



Contribution

Conclusion



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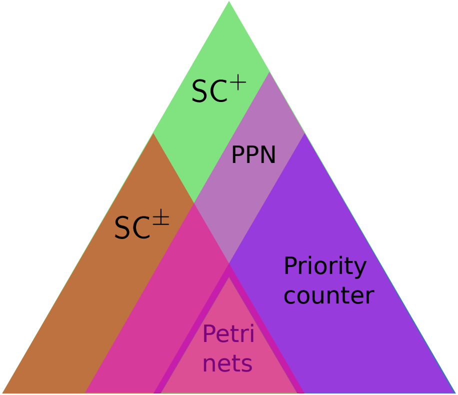

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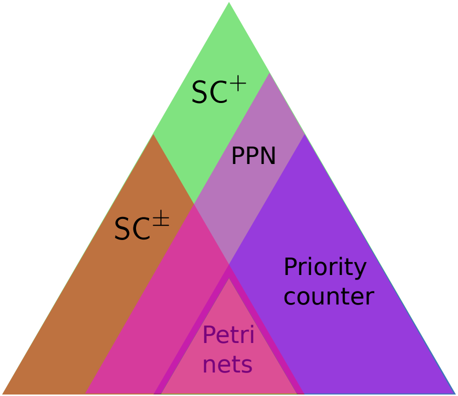

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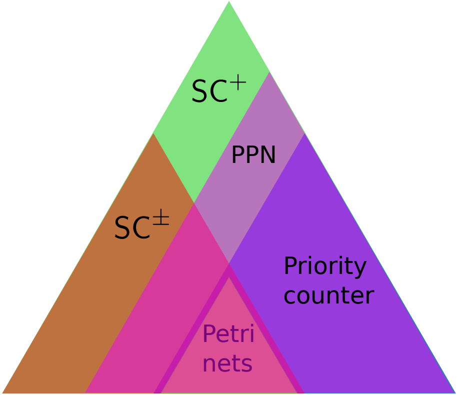

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