Semilinearity and Context-Freeness of Languages Accepted by Valence Automata

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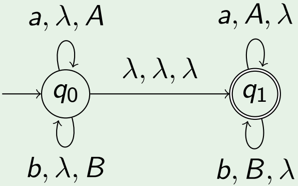
MFCS 2013

Example (Pushdown automaton)



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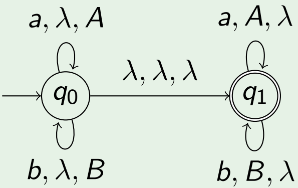

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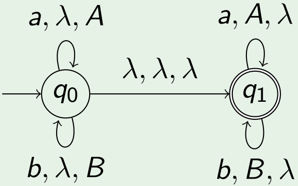

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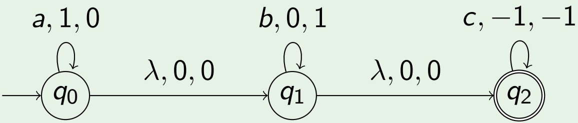

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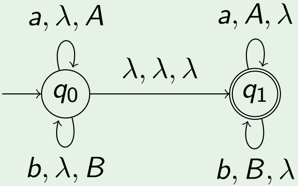

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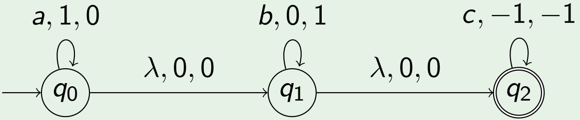
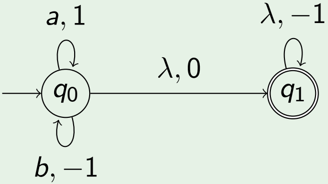


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



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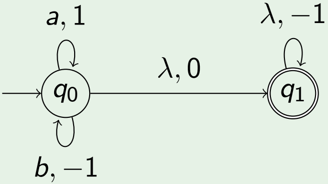

$$L = \{w \in \{a,b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for any prefix } p \text{ of } w\}$$

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

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Each storage mechanism consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \to S$

Model	States	Operations
Pushdown automata	$S = \Gamma^*$	$\mathrm{push}_a : w \mapsto wa,\ a \in \Gamma$ $\mathrm{pop}_a : wa \mapsto w,\ a \in \Gamma$

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Observation

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \mathrm{id}$$

Definition

A *monoid* is

- a set M together with
- an associative binary operation $\cdot : M \times M \to M$ and
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Storage mechanisms as monoids

- Let S be a set of states and $\alpha_1, \ldots, \alpha_n : S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid M.
- The identity map is the neutral element of M.
- M is a decription of the storage mechanism.

Valence automata

Common generalization: Valence Automata

Valence automaton over M:

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Language class

$\mathsf{VA}(M)$ languages accepted by valence automata over M.

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Definition (Graph products)

Let $\Gamma = (V, E)$ be a simple graph and M_v a monoid for each $v \in V$ with a presentation (A_v, R_v). Then the graph product $\mathbb{M}(\Gamma, (M_v)_{v \in V})$ is given by

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Intuition:

- $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ consists of sequences of elements in $\bigcup_v M_v$
- elements in the same monoid M_v are multiplied as defined in M_v
- an edge between v and w means elements from M_v and M_w commute

Specialization: Monoids defined by graphs

Notation

- $\mathbb{B}$: monoid for partially blind counter, $\mathbb{B} = \{a, \bar{a}\}^* / \{a\bar{a} = 1\}$.
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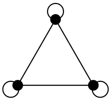
Monoids $\mathbb{M}\Gamma$

To each graph Γ, we associate the monoid $\mathbb{M}\Gamma$:

- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ is the corresponding graph product

Examples



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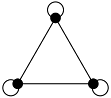

$\mathbb{Z}^3$

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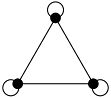

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Blind multicounter

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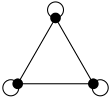

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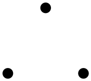

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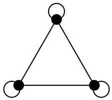

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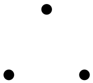

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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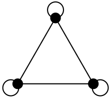

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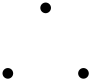

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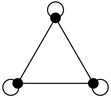

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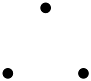

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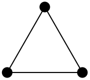

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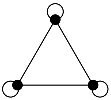

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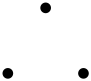

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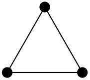

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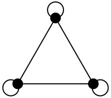

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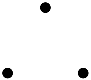

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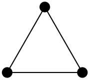

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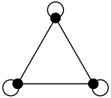

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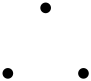

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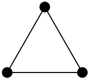

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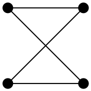

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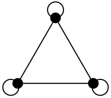

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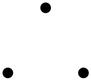

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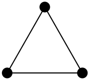

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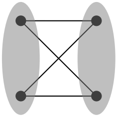

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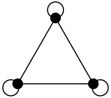

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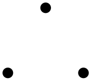

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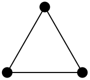

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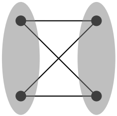

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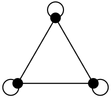

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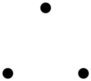

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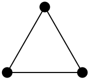

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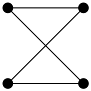

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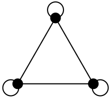
Infinite tape (TM)

Examples



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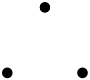

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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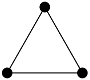

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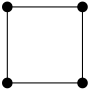

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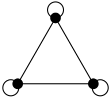
Infinite tape (TM)

Examples



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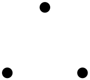

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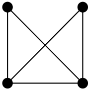

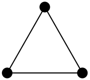

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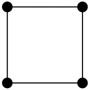

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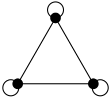

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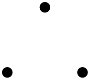

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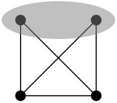

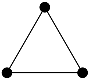

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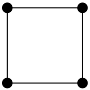

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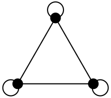

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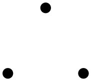

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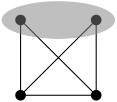

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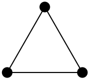

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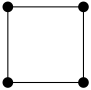

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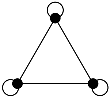

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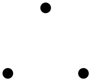

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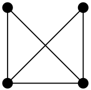

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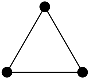
Pushdown + partially blind counters



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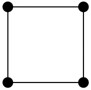

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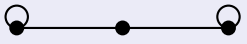
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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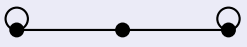
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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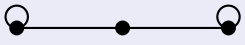
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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- $\mathsf{VA}(\mathbb{B}) \subseteq \mathsf{CF}$
- $M \mapsto M \times \mathbb{Z}$, $(M, M') \mapsto M * M'$ preserve semilinearity

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Definition

A monoid with the above property is called *FRI-monoid*.

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- Context-freeness is preserved by products $M *_F M'$, F a finite subgroup with $1 \in F$.

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Theorem

Let $\Gamma = (V, E)$ with $\mathsf{J}(M_v) \neq \{1\}$ for any $v \in V$. $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ is context-free iff

1. *for each $v \in V$, M_v is context-free,*
2. *if M_v and M_w are not FRI-monoids and $v \neq w$, then $\{v, w\} \notin E$,*
3. *if M_v is not an FRI-monoid, M_u and M_w are FRI-monoids and $\{v, u\}, \{v, w\} \in E$, then $\{u, w\} \in E$, and*
4. *the graph Γ is chordal.*

- $M \times M'$ context-free iff one factor is FRI and one is context-free.
- Context-freeness is preserved by products $M *_F M'$, F a finite subgroup with $1 \in F$.
- For induced cycles of finite groups, relied on Lohrey and Sénizergues' result.

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- Boolean closure