Knapsack in Graph Groups, HNN-Extensions and Amalgamated Products

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Equations and formal languages in algebra
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Theorem

For every virtually special group, compressed knapsack is in NP.

- virtually special: finite extension of a subgroup of a right-angled Artin group
- compressed knapsack: equation $g_1^{x_1} \cdots g_k^{x_k} = g$, where $g_1, \ldots, g_k, g$ are given by SLPs over $\Sigma \cup \Sigma^{-1}$.

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Groups of the form $\mathbb{G}(A, I)$ are called *right-angled Artin group*.

Semilinear sets

- A subset of $\mathbb{N}^k$ of the form

$$L = \left\{ v_0 + \sum_{i=1}^{n} x_i v_i \;\middle|\; x_1, \ldots, x_n \in \mathbb{N} \right\}$$

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Theorem (Ginsburg-Spanier 1966)

A set is semilinear if and only if it is first-order definable in $(\mathbb{N}, +, \geqslant, 0)$.

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Equivalence is effective → decidability

Theorem

Let $u_1, u_2, \ldots, u_n \in \mathbb{G}(A, I)\setminus\{1\}$, $v_0, v_1, \ldots, v_n \in \mathbb{G}(A, I)$ *and let* $x_1, \ldots, x_n$ *be variables ranging over* $\mathbb{N}$. *Then, the set of solutions of the exponent equation*

$$v_0 u_1^{x_1} v_1 u_2^{x_2} v_2 \cdots u_n^{x_n} v_n = 1$$

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- Lohrey and Schleimer (2007): compressed word problem for each right-angled Artin group in P.

Trace monoids

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- We consider $\mathbb{M}(A^{\pm 1}, I^{\pm 1})$, where

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- A trace t is *irreducible* if there is no decomposition $t = [uaa^{-1}v]_I$ for $a \in A^{\pm 1}, u, v \in (A^{\pm 1})^*$.

We call a trace t *connected* if there is no factorization $t = uv$ with $u \neq 1 \neq v$ and uIv.

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Lemma

Fix the alphabet A. Let $p, q, u, v, s, t \in \mathbb{M}(A, I)$ with $u \neq 1$ and $v \neq 1$ connected. Then the set

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid pu^x s = qv^y t\}$$

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- Techniques from recognizable trace languages:
- Construct finite automaton for $[pu^* s]_I \cap [qv^* t]_I$.

Levi's Lemma

Lemma

Let $u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{M}(A, I)$. *Then* $u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$ *if and only if there exist* $w_{i,j} \in \mathbb{M}(A, I)$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ *such that*

- $u_i = w_{i,1} w_{i,2} \cdots w_{i,n}$ *for every* $1 \leqslant i \leqslant m$,
- $v_j = w_{1,j} w_{2,j} \cdots w_{m,j}$ *for every* $1 \leqslant j \leqslant n$, *and*
- $(w_{i,j}, w_{k,\ell}) \in I$ *if* $1 \leqslant i < k \leqslant m$ *and* $n \geqslant j > \ell \geqslant 1$.

v_n	$w_{1,n}$	$w_{2,n}$	$w_{3,n}$	$\ldots$	$w_{m,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_3	$w_{1,3}$	$w_{2,3}$	$w_{3,3}$	$\ldots$	$w_{m,3}$
v_2	$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$\ldots$	$w_{m,2}$
v_1	$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$\ldots$	$w_{m,1}$
	u_1	u_2	u_3	$\ldots$	u_m

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Let $u_1, u_2, \ldots, u_n \in \mathrm{IRR}(A^{\pm 1}, I)$ be irreducible traces.
The sequence $u_1, u_2, \ldots, u_n$ is *I-freely reducible* if it can be reduced to the empty sequence ε by the following rules:

- $u_i, u_j \to u_j, u_i$ if $u_i I u_j$
- $u_i, u_j \to \varepsilon$ if $u_i = u_j^{-1}$ in $\mathbb{G}(A, I)$
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Lemma

Let $n \geqslant 2$ and $u_1, u_2, \ldots, u_n \in \mathrm{IRR}(A^{\pm 1}, I)$. If $u_1 u_2 \cdots u_n = 1$ in $\mathbb{G}(A, I)$, then there exist factorizations $u_i = u_{i,1} \cdots u_{i,k_i}$ such that the sequence

$$u_{1,1}, \ldots, u_{1,k_1},\ u_{2,1}, \ldots, u_{2,k_2},\ \ldots, u_{n,1}, \ldots, u_{n,k_n}$$

is I-freely reducible. Moreover, $\sum_{i=1}^{n} k_i \leqslant 2^n - 2$.

Lemma

Let $u^x = y_1 \cdots y_m$ be an equation where u is a concrete connected trace. It is equivalent to a disjunction of statements

$$\exists x_1, \ldots, x_m \geqslant 0\colon \quad x = \sum_{i=1}^{m} x_i + c \quad \wedge \quad \bigwedge_{i=1}^{m} y_i = p_i u^{x_i} s_i,$$

where

- *p_i, s_i are concrete traces of length polynomial in m and $|u|$*
- *c is a concrete number, polynomial in m*

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- Consider $v_0 \cdot u_1^{x_1} \cdot v_1 \cdot u_2^{x_2} \cdot v_2 \cdots u_n^{x_n} \cdot v_n = 1$
- By preprocessing, all factors $u_1^{x_1}, u_2^{x_2}, \ldots, u_n^{x_n}, v_0, \ldots, v_n$ are irreducible, connected

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- Guess which x_i are positive $\to$ eliminate commutation relations

- Consider $v_0 \cdot u_1^{x_1} \cdot v_1 \cdot u_2^{x_2} \cdot v_2 \cdots u_n^{x_n} \cdot v_n = 1$
- By preprocessing, all factors $u_1^{x_1}, u_2^{x_2}, \dots, u_n^{x_n}, v_0, \dots, v_n$ are irreducible, connected
- Apply exponential refinement to obtain I-freely reducible sequence.
- Consider all possible refinements and all possible reduction sequences.
- We obtain a disjunction of statements:

  (c) $y_{i,j} = y_{k,l}^{-1}$
- Replace $z_{k,l}$ by concrete traces.
- Replace $u_i^{x_i} = y_{i,1} \cdots y_{i,k_i}$

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- Result of von zur Gathen and Sieveking (1978) yields a small solution.

Transfer results

Theorem

The class of groups with knapsack in NP *is closed under*

- *Taking finite extensions*
- *HNN-extensions over finite associated subgroups*
- *Amalgamated products with finite identified groups*

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- Saturation procedure that successively adds transitions to automaton
- Choose suitable class of automata such that adding transitions still leads to knapsack instances: knapsack automata.