Effectively Regular Downward Closures

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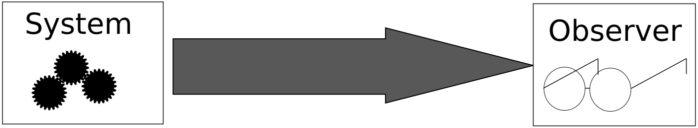
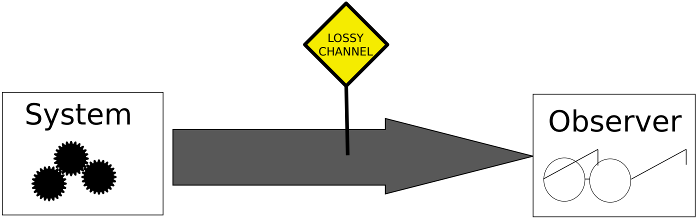
LSV Cachan, 28 October 2014


System
Observer


System
LOSSY
CHANNEL
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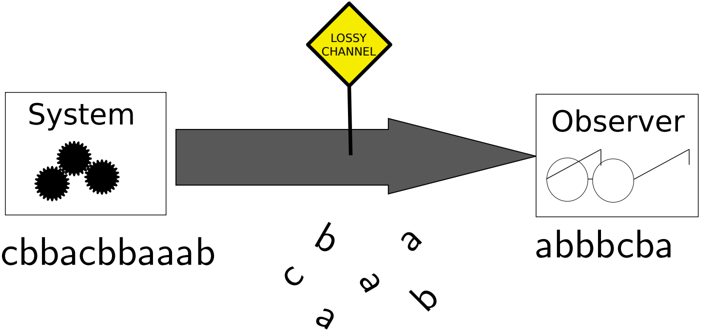

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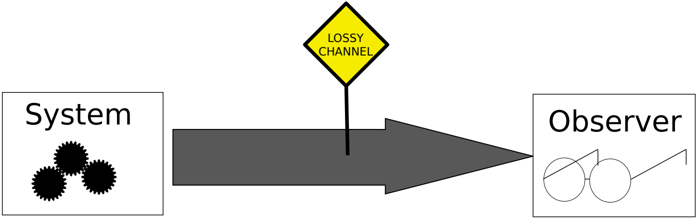

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Downward Closures

- $u \preccurlyeq v$: u is a subsequence of v
- $L{\downarrow} = \{u \in X^* \mid \exists v \in L\colon u \preccurlyeq v\}$
- Oberver sees precisely $L{\downarrow}$

Downward Closures

Theorem (Higman/Haines)

For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.

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Problem

- Finite automaton for $L\downarrow$ exists for every L.
- How can we compute it?

State of the art

Very few known techniques.

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Theorem (Habermehl, Meyer, Wimmel 2010)

Downward closures are computable for Petri net languages.

Stacked counter automata

A storage mechanism M consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n \colon S \to S$
- Initial state: $s_0 \in S$
- Final states: $F \subseteq S$

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- Operations: increment, decrement, zero test
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Trivial mechanism

Consists of one state and no operations.

$C(M)$: Adding a blind counter

- States: (s, z), s an old state, $z \in \mathbb{Z}$.
- Operations: old operations; increment, decrement for counter
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$S(M)$: Building stacks

- States: sequences $\square c_1 \square c_2 \square \cdots \square c_n$, c_i old states
- Operations: push separator, pop if empty, manipulate topmost entry
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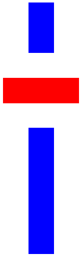
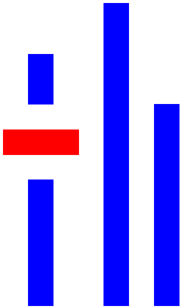
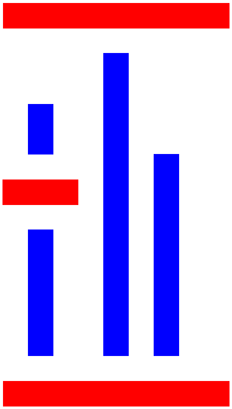
Stacked counters

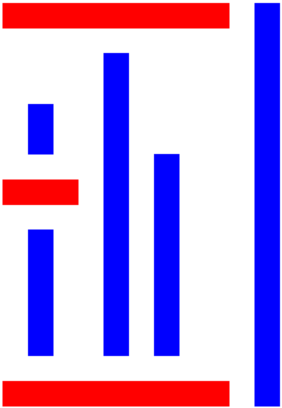
Mechanisms obtained from the trivial one by

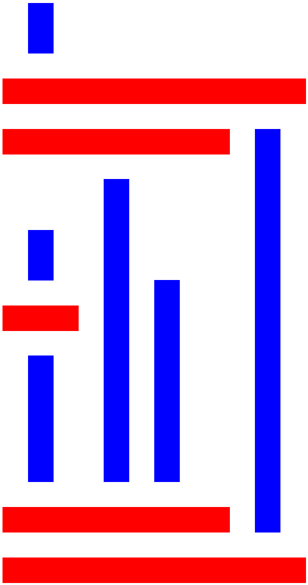
- adding blind counters,
- building stacks.











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- Generalize both pushdown automata and blind counter automata

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Theorem (Z. 2014)

Downward closures are computable for stacked counter automata.

Expressiveness

Algebraic extensions

Let $\mathcal{C}$ be a language class. A *$\mathcal{C}$-grammar* G consists of

- Nonterminals N, terminals T, start symbol $S \in N$
- Productions $A \to L$ with $L \subseteq (N \cup T)^*$, $L \in \mathcal{C}$

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Example

$\mathrm{Alg}(\mathrm{FIN}) = \mathrm{Alg}(\mathrm{REG}) = \mathrm{CF}$

Definition

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Semilinear constraints

Let $\mathcal{C}$ be a language class. $\mathsf{SLI}(\mathcal{C})$ denotes the class of languages

$$h(L \cap \Psi^{-1}(S))$$

for some $L \in \mathcal{C}$, a homomorphism h and a semilinear set S.

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$h(a^*bc^* \cap \Psi^{-1}(b + (a+c)^{\oplus})) = \{a^nba^n \mid n \geqslant 0\},\ h\colon a, c \mapsto a,\ b \mapsto b.$

A hierarchy of language classes

Hierarchy

$$F_0 = \text{finite languages},$$

$$G_i = \mathsf{Alg}(F_i), \qquad F_{i+1} = \mathsf{SLI}(G_i), \qquad F = \bigcup_{i \geqslant 0} F_i.$$

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Corollary

Stacked counter automata accept precisely the languages in F.

Ingredient I

van Leeuwen proved a stronger statement:

Theorem (van Leeuwen 1978)

If $\mathcal{C}$ is closed under regular intersections:
Downward closures computable for $\mathcal{C} \implies$ computable for $\mathsf{Alg}(\mathcal{C})$.

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Consequence

Algorithm for $\mathsf{F}_i \implies$ Algorithm for $\mathsf{G}_i = \mathrm{Alg}(\mathsf{F}_i)$.

van Leeuwen's Algorithm

- Replace each production $A \to L$ with $A \to L{\downarrow}$.

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van Leeuwen's Algorithm

- Replace each production $A \to L$ with $A \to L\!\downarrow$.

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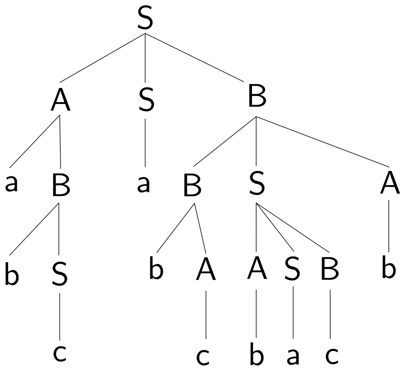

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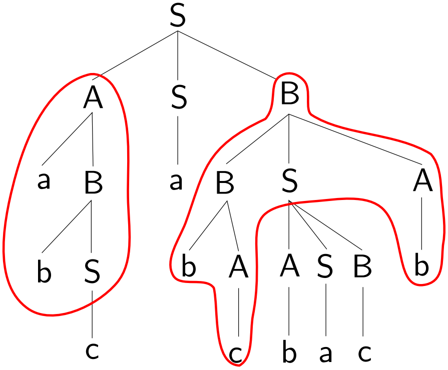

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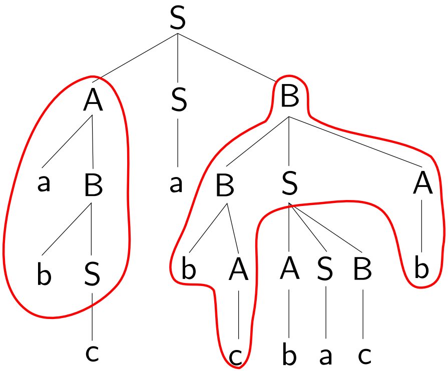

Algorithm

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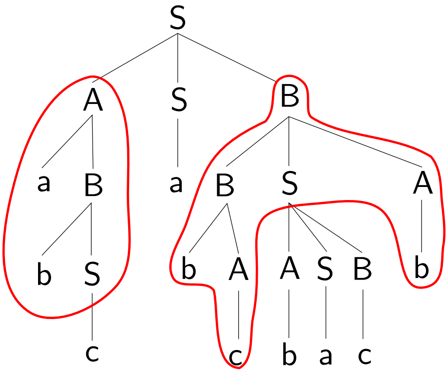
van Leeuwen's Algorithm



Algorithm

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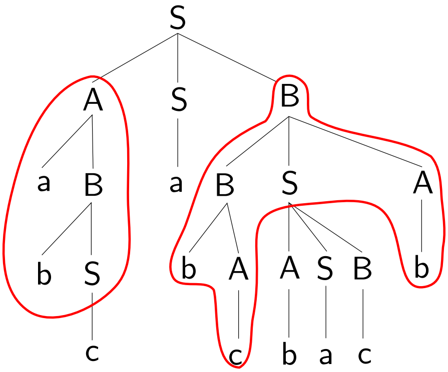
  - S is now terminal
- G_A has fewer nonterminals

van Leeuwen's Algorithm



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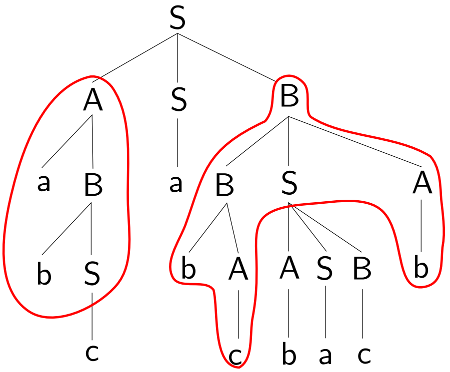
- Compute $\mathsf{L}(G_A)\downarrow$

van Leeuwen's Algorithm



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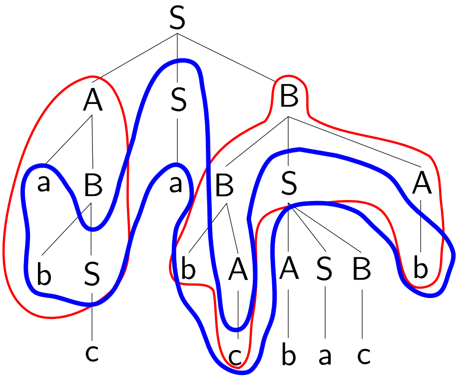

S → abSSbcSb

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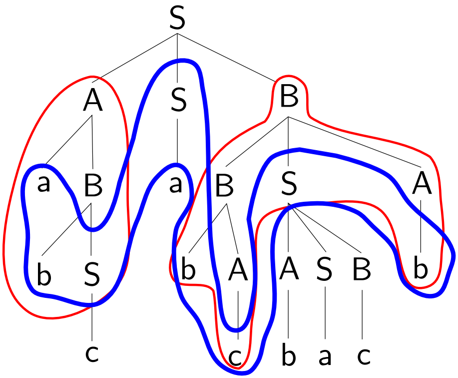

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- Compute $\mathrm{L}(G_A)\!\downarrow$
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- Resulting grammar has one nonterminal

Ingredient II

$$F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F$$

Problem

- Computability preserved by $\mathrm{Alg}(\cdot)$

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Theorem (Parikh)

For context-free L, $\Psi(L)$ is semilinear.

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Theorem (Parikh)

For context-free L, $\Psi(L)$ is semilinear.

$$\Psi(L) = \bigcup_{i=1}^{n} \mu_i + F_i^{\oplus}$$

- μ_i: constant vector
- F_i: set of period vectors

Example

$$L = (ab)^*(ca^* \cup db^*)$$

Parikh image: $c + \{a+b, a\}^{\oplus} \quad \cup \quad d + \{a+b, b\}^{\oplus}$.

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$$K = \alpha(\mu ab)^* c(\nu a)^* \quad \cup \quad \beta(\sigma ab)^* d(\tau b)^*$$

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- New language in the same class
- Additional symbols encode decomposition of Parikh image into constant and period vectors
- Adding period vectors by inserting at designated positions

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Let $\mathcal{C}$ be a language class and $L \in \mathcal{C}$, $L \subseteq X^*$. A Parikh annotation for L in $\mathcal{C}$ is a tuple $(K, C, P, (P_c)_{c \in C}, \varphi)$, where

- $K \subseteq C(X \cup P)^*$ is in $\mathcal{C}$,
- $\pi_X(K) = L$,
- $\Psi(\pi_X(w)) = \varphi(\pi_{C \cup P}(w))$ for each $w \in K$,
- $\Psi(\pi_{C \cup P}(K)) = \bigcup_{c \in C} c + P_c^{\oplus}$.
- For $cw \in K$ and $\kappa \in P_c^{\oplus}$, there is a $v \in L$ with

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For each level of the hierarchy, one can construct Parikh annotations.

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Lemma

Let G be a reduced $\mathcal{C}$-grammar and $\psi\colon T^* \to \mathbb{Z}$ a morphism such that $\psi(w) = 0$ for every $w \in L(G)$. Then ψ extends uniquely to a G-compatible morphism $\psi\colon (N \cup T)^* \to \mathbb{Z}$.

Using Parikh annotations, one can show:

Corollary

Given $L \in \mathsf{G}_i$ and semilinear S, one can construct $L' \in \mathsf{G}_i$ with $L \cap \Psi^{-1}(S) \subseteq L' \subseteq (L \cap \Psi^{-1}(S))\!\downarrow$.

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- Select all words where adding period vectors leads into S
- Downward closed set of multisets of period vectors
- Recognizable by finite automaton

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Thank you for your attention!

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- Let L' be obtained from K by replacing every $x \in C \cup P$ by $a^{\varphi(x)(a)}$.
- Then $L' = \{a^n b^n a^n \mid n \geqslant 0\}$, which is not context-free.