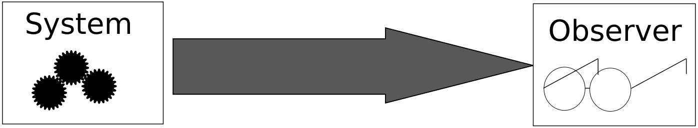
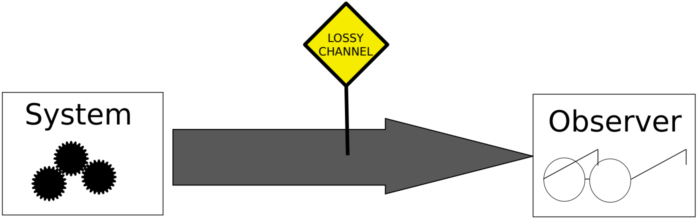
# An approach to computing downward closures

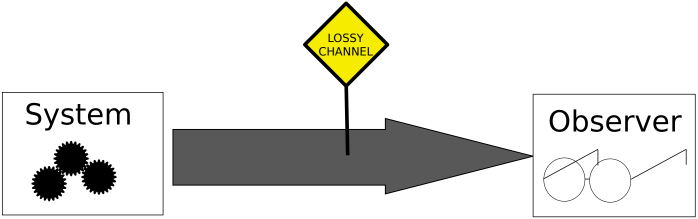
Georg Zetzsche

Technische Universität Kaiserslautern

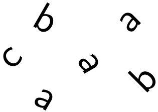
ICALP 2015


System
Observer


System
LOSSY
CHANNEL
Observer

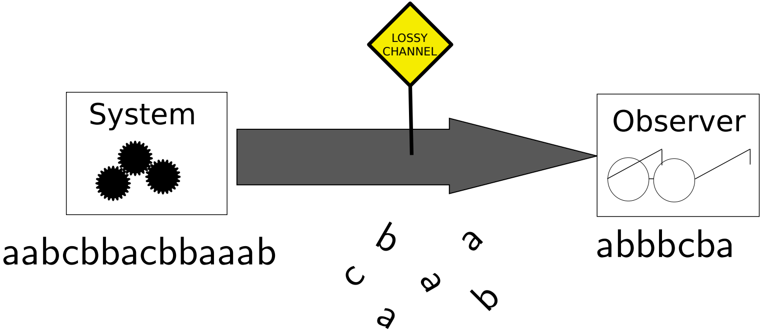



aabcbbacbbaaab



abbbcba

aabcbbacbbaaab




$\underline{a}ab\underline{c}bb\underline{a}cb\underline{b}a\underline{aab}$

## Downward Closures

- $u \preccurlyeq v$: $u$ is a subsequence of $v$
- $L{\downarrow} = \{u \in X^* \mid \exists v \in L \colon u \preccurlyeq v\}$
- Observer sees precisely $L{\downarrow}$

# Downward Closures

**Theorem (Higman/Haines)**

*For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.*

# Downward Closures

## Theorem (Higman/Haines)

*For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.*

## Applications

Given an automaton for $L{\downarrow}$, many things are decidable:

# Downward Closures

**Theorem (Higman/Haines)**

*For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.*

**Applications**

Given an automaton for $L{\downarrow}$, many things are decidable:

- Inclusion of behavior under lossy observation ($K{\downarrow} \subseteq L{\downarrow}$)
  Ordinary inclusion almost always undecidable!

# Downward Closures

## Theorem (Higman/Haines)

*For every language $L \subseteq X^*$, $L{\downarrow}$ is regular.*

## Applications

Given an automaton for $L{\downarrow}$, many things are decidable:

- Inclusion of behavior under lossy observation ($K{\downarrow} \subseteq L{\downarrow}$)
  Ordinary inclusion almost always undecidable!
- Which actions occur arbitrarily often? ($a^* \subseteq L{\downarrow}$)

# Downward Closures

## Theorem (Higman/Haines)

*For every language $L \subseteq X^*$, $L\downarrow$ is regular.*

## Applications

Given an automaton for $L\downarrow$, many things are decidable:

- Inclusion of behavior under lossy observation ($K\downarrow \subseteq L\downarrow$)
  Ordinary inclusion almost always undecidable!
- Which actions occur arbitrarily often? ($a^* \subseteq L\downarrow$)
- Is $a$ ever executed after $b$? ($ab \in L\downarrow$)

# Downward Closures

**Theorem (Higman/Haines)**

*For every language $L \subseteq X^*$, $L\downarrow$ is regular.*

**Applications**

Given an automaton for $L\downarrow$, many things are decidable:

- Inclusion of behavior under lossy observation ($K\downarrow \subseteq L\downarrow$)
  Ordinary inclusion almost always undecidable!
- Which actions occur arbitrarily often? ($a^* \subseteq L\downarrow$)
- Is $a$ ever executed after $b$? ($ab \in L\downarrow$)
- Can the system run arbitrarily long? ($L\downarrow$ infinite)

# Downward Closures

## Theorem (Higman/Haines)

*For every language $L \subseteq X^*$, $L\downarrow$ is regular.*

## Applications

Given an automaton for $L\downarrow$, many things are decidable:

- Inclusion of behavior under lossy observation ($K\downarrow \subseteq L\downarrow$)
  Ordinary inclusion almost always undecidable!
- Which actions occur arbitrarily often? ($a^* \subseteq L\downarrow$)
- Is $a$ ever executed after $b$? ($ab \in L\downarrow$)
- Can the system run arbitrarily long? ($L\downarrow$ infinite)

## Problem

- Finite automaton for $L\downarrow$ exists for every $L$.
- How can we compute it?

# Negative results

**Theorem (Gruber, Holzer, Kutrib 2009)**

*Downward closures are not computable when infinity or emptiness are undecidable.*

**Theorem (Mayr 2003)**

*The reachability set of lossy channel systems is not computable.*

# Positive results

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

**Theorem (Abdulla, Boasson, Bouajjani, ICALP 2001)**

*Downward closures are computable for context-free FIFO rewriting systems/0L-systems.*

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

**Theorem (Abdulla, Boasson, Bouajjani, ICALP 2001)**

*Downward closures are computable for context-free FIFO rewriting systems/0L-systems.*

Context-free rules $A \rightarrow w$, applied as: $Au \Rightarrow uw$

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

**Theorem (Abdulla, Boasson, Bouajjani, ICALP 2001)**

*Downward closures are computable for context-free FIFO rewriting systems/0L-systems.*

Context-free rules $A \to w$, applied as: $Au \Rightarrow uw$

**Theorem (Habermehl, Meyer, Wimmel, ICALP 2010)**

*Downward closures are computable for Petri net languages.*

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

**Theorem (Abdulla, Boasson, Bouajjani, ICALP 2001)**

*Downward closures are computable for context-free FIFO rewriting systems/0L-systems.*

Context-free rules $A \to w$, applied as: $Au \Rightarrow uw$

**Theorem (Habermehl, Meyer, Wimmel, ICALP 2010)**

*Downward closures are computable for Petri net languages.*

**Theorem (Z., STACS 2015)**

*Downward closures are computable for stacked counter automata.*

# Positive results

**Theorem (van Leeuwen 1978/Courcelle 1991)**

*Downward closures are computable for context-free languages.*

**Theorem (Abdulla, Boasson, Bouajjani, ICALP 2001)**

*Downward closures are computable for context-free FIFO rewriting systems/0L-systems.*

Context-free rules $A \to w$, applied as: $Au \Rightarrow uw$

**Theorem (Habermehl, Meyer, Wimmel, ICALP 2010)**

*Downward closures are computable for Petri net languages.*

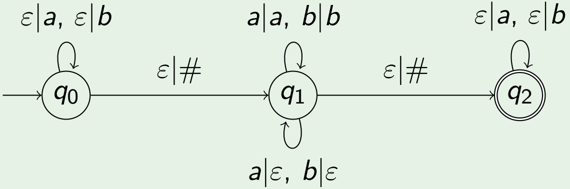
**Theorem (Z., STACS 2015)**

*Downward closures are computable for stacked counter automata.*

- Weak form of stack nesting
- Adding Counters

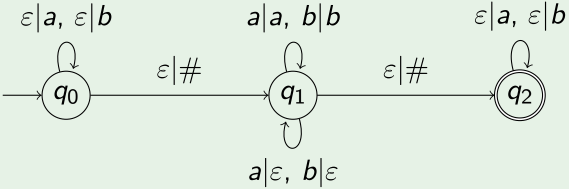
# A general approach

## Example (Transducer)



# A general approach

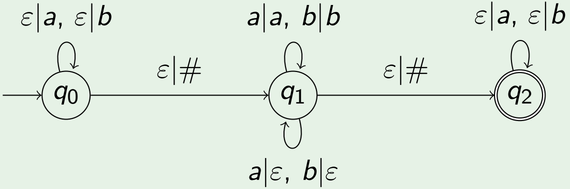
## Example (Transducer)




$$T(A) = \{(x, u\#v\#w) \mid u, v, w, x \in \{a, b\}^*,\ v \leqslant x\}$$

# A general approach

## Example (Transducer)




$$T(A) = \{(x, u\#v\#w) \mid u, v, w, x \in \{a, b\}^*,\ \ v \leqslant x\}$$

## Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{y \in X^* \mid \exists x \in L : (x, y) \in T\}$$

### Definition

$\mathcal{C}$ is a *full trio* if $LR \in \mathcal{C}$ for each $L \in \mathcal{C}$ and rational transduction $R$.

### Theorem

*If $\mathcal{C}$ is a full trio, then downward closures are computable for $\mathcal{C}$ if and only if the simultaneous unboundedness problem is decidable:*

*Given* A language $L \subseteq a_1^* \cdots a_n^*$ in $\mathcal{C}$

*Question* Is $a_1^* \cdots a_n^*$ included in $L{\downarrow}$?

**Theorem (Jullien 1969, Abdulla et. al. 2004)**

*Every language $L\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages"

Theorem (Jullien 1969, Abdulla et. al. 2004)

*Every language $L\!\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages" $\leftarrow$ Ideal decomposition!

## Theorem (Jullien 1969, Abdulla et. al. 2004)

*Every language $L\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages" $\leftarrow$ Ideal decomposition!

## Algorithm

Suppose $L \subseteq X^*$ is given.
Enumerate simple regular languages $R$.
Decide whether $L\downarrow = R$:

### Theorem (Jullien 1969, Abdulla et. al. 2004)

*Every language $L\!\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages" $\leftarrow$ Ideal decomposition!

### Algorithm

Suppose $L \subseteq X^*$ is given.
Enumerate simple regular languages $R$.
Decide whether $L\!\downarrow = R$:

- $L\!\downarrow \subseteq R$ iff $L\!\downarrow \cap (X^* \setminus R) = \emptyset \rightsquigarrow$ emptiness.

## Theorem (Jullien 1969, Abdulla et. al. 2004)

*Every language $L\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages" $\leftarrow$ Ideal decomposition!

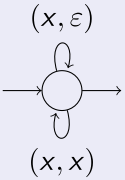
## Algorithm

Suppose $L \subseteq X^*$ is given.
Enumerate simple regular languages $R$.
Decide whether $L\downarrow = R$:

- $L\downarrow \subseteq R$ iff $L\downarrow \cap (X^* \setminus R) = \emptyset \rightsquigarrow$ emptiness.

## Observation

$L\downarrow$ is in $\mathcal{C}$:



## Theorem (Jullien 1969, Abdulla et. al. 2004)

*Every language $L\!\downarrow$ can be written as a finite union of sets of the form*

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

*where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.*

"Simple Regular Languages" $\longleftarrow$ Ideal decomposition!

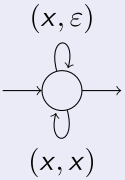
## Algorithm

Suppose $L \subseteq X^*$ is given.
Enumerate simple regular languages $R$.
Decide whether $L\!\downarrow = R$:

- $L\!\downarrow \subseteq R$ iff $L\!\downarrow \cap (X^* \setminus R) = \emptyset \rightsquigarrow$ emptiness.
- $R \subseteq L\!\downarrow \rightsquigarrow Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L\!\downarrow$

## Observation

$L\!\downarrow$ is in $\mathcal{C}$:



## Observation

- It suffices to check whether $Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$.

## Observation

- It suffices to check whether $Y_0^*\{x_1, \varepsilon\}Y_1^* \cdots \{x_n, \varepsilon\}Y_n^* \subseteq L{\downarrow}$.
- $L{\downarrow}$ includes $\{a, b, c\}^*$ if and only if it contains $(abc)^*$.

## Observation

- It suffices to check whether $Y_0^*\{x_1, \varepsilon\}Y_1^* \cdots \{x_n, \varepsilon\}Y_n^* \subseteq L{\downarrow}$.
- $L{\downarrow}$ includes $\{a, b, c\}^*$ if and only if it contains $(abc)^*$.

$$abc\ abc\ abc\ abc\ abc$$

## Observation

- It suffices to check whether $Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$.
- $L{\downarrow}$ includes $\{a,b,c\}^*$ if and only if it contains $(abc)^*$.

$$abc\ abc\ abc\ abc\ abc$$

$$bacca$$

## Observation

- It suffices to check whether $Y_0^*\{x_1, \varepsilon\}Y_1^* \cdots \{x_n, \varepsilon\}Y_n^* \subseteq L{\downarrow}$.
- $L{\downarrow}$ includes $\{a, b, c\}^*$ if and only if it contains $(abc)^*$.

$$abc\ abc\ abc\ abc\ abc$$

$$bacca$$

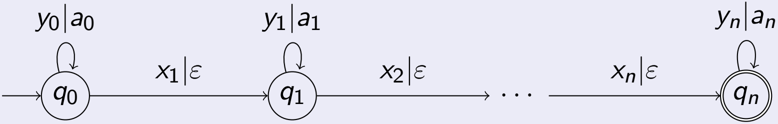
## Observation

- It suffices to check whether $Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L\!\downarrow$.
- $L\!\downarrow$ includes $\{a,b,c\}^*$ if and only if it contains $(abc)^*$.

$$abc\ abc\ abc\ abc\ abc$$

$$bacca$$

## Transduction $T$




$y_i$: word containing each letter of $Y_i$ once.

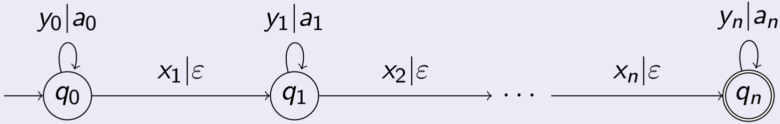
## Observation

- It suffices to check whether $Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$.
- $L{\downarrow}$ includes $\{a,b,c\}^*$ if and only if it contains $(abc)^*$.

$$abc\ abc\ abc\ abc\ abc$$

$$bacca$$

## Transduction $T$




$y_i$: word containing each letter of $Y_i$ once. Then:

$$T(L{\downarrow}){\downarrow} = a_0^*\cdots a_n^* \quad \text{iff} \quad Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$$

# New algorithm for each known positive case

Context-free grammars and stacked counter automata:

**Corollary**

*If $\mathcal{C}$ is a full trio and has effectively semilinear Parikh images, then downward closures are computable for $\mathcal{C}$.*

# New algorithm for each known positive case

Context-free grammars and stacked counter automata:

**Corollary**

*If $\mathcal{C}$ is a full trio and has effectively semilinear Parikh images, then downward closures are computable for $\mathcal{C}$.*

Petri net languages $\rightsquigarrow$ boundedness with one inhibitor arc (Czerwiński, Martens 2015), decidable by (Bonnet et. al. 2012)

# New algorithm for each known positive case

Context-free grammars and stacked counter automata:

**Corollary**

*If $\mathcal{C}$ is a full trio and has effectively semilinear Parikh images, then downward closures are computable for $\mathcal{C}$.*

Petri net languages $\rightsquigarrow$ boundedness with one inhibitor arc (Czerwiński, Martens 2015), decidable by (Bonnet et. al. 2012)

**Theorem**

*Downward closures are computable for matrix languages.*

Natural generalization of context-free and Petri net languages.

# New algorithm for each known positive case

Context-free grammars and stacked counter automata:

**Corollary**

*If $\mathcal{C}$ is a full trio and has effectively semilinear Parikh images, then downward closures are computable for $\mathcal{C}$.*

Petri net languages $\rightsquigarrow$ boundedness with one inhibitor arc (Czerwiński, Martens 2015), decidable by (Bonnet et. al. 2012)

**Theorem**

*Downward closures are computable for matrix languages.*

Natural generalization of context-free and Petri net languages.

**Theorem**

*Downward closures are computable for indexed languages.*

(Generalize 0L-systems)

# Indexed Grammars

## Indexed Grammars

Idea: Each nonterminal carries a stack.

# Indexed Grammars

## Indexed Grammars

Idea: Each nonterminal carries a stack.

Tuple $G = (N, T, I, P, S)$, where

- $N, T, I$ are nonterminal, terminal, index alphabet,
- $S \in N$ start symbol

# Indexed Grammars

## Indexed Grammars

Idea: Each nonterminal carries a stack.

Tuple $G = (N, T, I, P, S)$, where

- $N, T, I$ are nonterminal, terminal, index alphabet,
- $S \in N$ start symbol
- Productions $P$ of the form
  - $A \to Bf$, push index ($f \in I$)
  - $Af \to B$, pop index ($f \in I$)
  - $A \to uBv$, generate terminals ($u, v \in T^*$)
  - $A \to BC$, split and duplicate index word
  - $A \to w$, generate only terminals ($w \in T^*$)

# Indexed Grammars

## Indexed Grammars

Idea: Each nonterminal carries a stack.

Tuple $G = (N, T, I, P, S)$, where

- $N, T, I$ are nonterminal, terminal, index alphabet,
- $S \in N$ start symbol
- Productions $P$ of the form
  - $A \to Bf$, push index ($f \in I$)
  - $Af \to B$, pop index ($f \in I$)
  - $A \to uBv$, generate terminals ($u, v \in T^*$)
  - $A \to BC$, split and duplicate index word
  - $A \to w$, generate only terminals ($w \in T^*$)

$$\begin{array}{llll} S \to Sf, & S \to Sg, & S \to UU, & U \to \varepsilon, \\ Uf \to A, & Ug \to B, & A \to Ua, & B \to Ub. \end{array}$$

$$N = \{S, T, A, B\}, I = \{f, g\}, T = \{a, b\}.$$

# Indexed Grammars

## Indexed Grammars

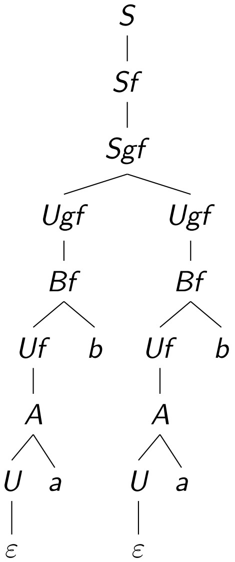
Idea: Each nonterminal carries a stack.

Tuple $G = (N, T, I, P, S)$, where

- $N, T, I$ are nonterminal, terminal, index alphabet,
- $S \in N$ start symbol
- Productions $P$ of the form
  - $A \to Bf$, push index ($f \in I$)
  - $Af \to B$, pop index ($f \in I$)
  - $A \to uBv$, generate terminals ($u, v \in T^*$)
  - $A \to BC$, split and duplicate index word
  - $A \to w$, generate only terminals ($w \in T^*$)

$$S \to Sf, \quad S \to Sg, \quad S \to UU, \quad U \to \varepsilon,$$
$$Uf \to A, \quad Ug \to B, \quad A \to Ua, \quad B \to Ub.$$

$$N = \{S, T, A, B\}, I = \{f, g\}, T = \{a, b\}.$$



# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^*b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

# Application to Indexed Languages

**No exact representation**

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L{\downarrow}$.

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\downarrow$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L{\downarrow}$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^*b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L{\downarrow}$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^*b^*$ intersect with $\{a^nb^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\downarrow$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$
  - the yields of such subtrees are unbounded in length

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\downarrow$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$
  - the yields of such subtrees are unbounded in length

## Step 1: Direct and indirect letters

For each subset $D \subseteq \{a_1, \ldots, a_n\}$, construct $G_D$

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\!\downarrow$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$
  - the yields of such subtrees are unbounded in length

## Step 1: Direct and indirect letters

For each subset $D \subseteq \{a_1, \ldots, a_n\}$, construct $G_D$:

- for $a_i \in D$, instead of deriving whole $a_i$-subtree, generate one $a_i$
- for $a_i \notin D$, derive only one of the $a_i$-subtrees

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\!\downarrow$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$
  - the yields of such subtrees are unbounded in length

## Step 1: Direct and indirect letters

For each subset $D \subseteq \{a_1, \ldots, a_n\}$, construct $G_D$:

- for $a_i \in D$, instead of deriving whole $a_i$-subtree, generate one $a_i$
- for $a_i \notin D$, derive only one of the $a_i$-subtrees $\leftarrow$ "indirect"

# Application to Indexed Languages

## No exact representation

Undeciable: Does $L \subseteq a^* b^*$ intersect with $\{a^n b^n \mid n \geqslant 0\}$?

Given: indexed grammar $G$ with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L{\downarrow}$.

## Observation

- Consider the derivations for $a_1^k \cdots a_n^k$, $k \geqslant 0$.
- For each $a_i$, at least one of the following holds:
  - there is an unbounded number subtrees with yield in $a_i^*$
  - the yields of such subtrees are unbounded in length

## Step 1: Direct and indirect letters

For each subset $D \subseteq \{a_1, \ldots, a_n\}$, construct $G_D$:

- for $a_i \in D$, instead of deriving whole $a_i$-subtree, generate one $a_i$
- for $a_i \notin D$, derive only one of the $a_i$-subtrees $\leftarrow$ “indirect”

Then, $a_1^* \cdots a_n^* \subseteq L(G){\downarrow}$ iff $a_1^* \cdots a_n^* \subseteq L(G_D){\downarrow}$ for some $D$.

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$
- Then the nonterminals have pairwise distinct intervals

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$
- Then the nonterminals have pairwise distinct intervals

$\Rightarrow$ Bounded number of occurrences

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$
- Then the nonterminals have pairwise distinct intervals

$\Rightarrow$ Bounded number of occurrences

Therefore: Replace these subtrees with linear ones

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$
- Then the nonterminals have pairwise distinct intervals

$\Rightarrow$ Bounded number of occurrences

Therefore: Replace these subtrees with linear ones

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Idea

Instead of unfolding $a_i$-subtree with root $Au$, $u \in I^*$, apply transducer to $u$

## Goal: bound nonterminal occurrences

Only obstacle: $a_i$-subtrees for indirect $a_i$

- Consider the interval $a_i^* \cdots a_j^*$ for each occurring nonterminal
- Suppose: no unfolding of $a_i$-subtrees, indirect $a_i$
- Then the nonterminals have pairwise distinct intervals

$\Rightarrow$ Bounded number of occurrences

Therefore: Replace these subtrees with linear ones

$$a_1 S_{(1,2)} a_2 a_2 T_{(3)} U_{(4)} a_5 V_{(5,8)} a_7 a_8 a_8 W_{(9)}$$

Indirect symbols: $\{a_3, a_4, a_9\}$

## Idea

Instead of unfolding $a_i$-subtree with root $Au$, $u \in I^*$, apply transducer to $u$
However: Precise simulation not possible

## Preserving $a_1^* \cdots a_n^* \subseteq L(G)\downarrow$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N} \cup \{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (Au, v) \in T\}$$
$$f_G(Au) = \sup\{|v| \mid v \in a_i^*,\ Au \Rightarrow_G^* v\}$$

## Preserving $a_1^* \cdots a_n^* \subseteq L(G)\downarrow$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N} \cup \{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (Au, v) \in T\}$$
$$f_G(Au) = \sup\{|v| \mid v \in a_i^*,\ Au \Rightarrow_G^* v\}$$

## Proposition

*For each indexed grammar $G$, one can construct a rational transduction $T$ with $f_T \approx f_G$.*

$f \approx g$: $f$ is unbounded on the same subsets as $g$
($\to$ regular cost functions)

## Preserving $a_1^* \cdots a_n^* \subseteq L(G)\downarrow$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N} \cup \{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (Au, v) \in T\}$$
$$f_G(Au) = \sup\{|v| \mid v \in a_i^*,\ Au \Rightarrow_G^* v\}$$

## Proposition

*For each indexed grammar $G$, one can construct a rational transduction $T$ with $f_T \approx f_G$.*

$f \approx g$: $f$ is unbounded on the same subsets as $g$
($\to$ regular cost functions)

## Step 2: Apply transducer

- Only one nonterminal occurrence for transducer

## Preserving $a_1^* \cdots a_n^* \subseteq L(G){\downarrow}$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N} \cup \{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (Au, v) \in T\}$$
$$f_G(Au) = \sup\{|v| \mid v \in a_i^*,\ Au \Rightarrow_G^* v\}$$

## Proposition

*For each indexed grammar $G$, one can construct a rational transduction $T$ with $f_T \approx f_G$.*

$f \approx g$: $f$ is unbounded on the same subsets as $g$
($\to$ regular cost functions)

## Step 2: Apply transducer

- Only one nonterminal occurrence for transducer

$\Rightarrow$ Bound on nonterminal occurrences, “breadth-bounded”

## Remaining problem

- Given: Breadth-bounded indexed grammar $G$, $L(G) \subseteq a_1^* \cdots a_n^*$
- Is $a_1^* \cdots a_n^*$ included in $L(G)\downarrow$?

## Remaining problem

- Given: Breadth-bounded indexed grammar $G$, $L(G) \subseteq a_1^* \cdots a_n^*$
- Is $a_1^* \cdots a_n^*$ included in $L(G){\downarrow}$?

Step 3:

## Proposition

*Breadth-bounded indexed grammars have effectively semilinear Parikh images.*

Thank you for your attention!