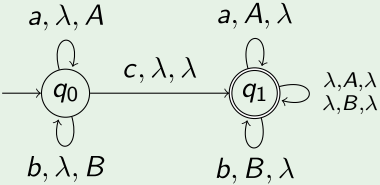
Silent Transitions in Automata with Storage

Georg Zetzsche

Technische Universität Kaiserslautern

ICALP 2013

Example (Pushdown automaton)



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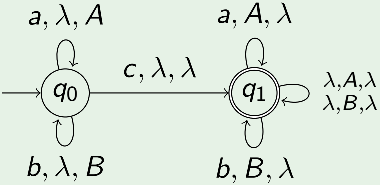

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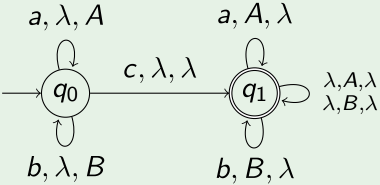

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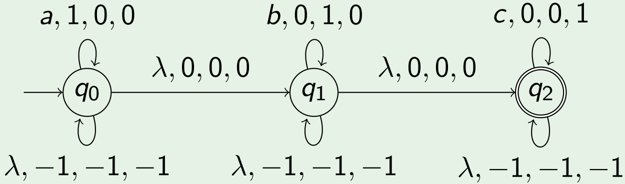

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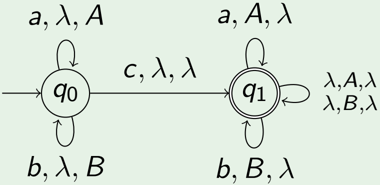

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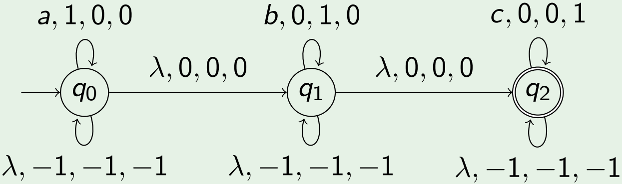
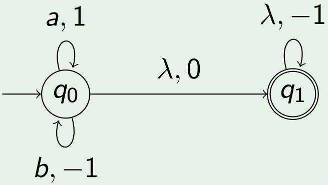


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



$$L = \{w \in \{a, b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for any prefix } p \text{ of } w\}$$

Silent Transitions

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- When can silent transitions be eliminated?
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Known so far

- Pushdown automata (Greibach 1965)
- Blind counter automata (Greibach 1978)
- Partially blind counter automata (Greibach 1978 / Jantzen 1979)

Valence automata

Definition

A *monoid* is a set M together with a binary associative operation and neutral element $1 \in M$.

Common generalization: Valence Automata

Valence automaton over M:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.

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$\mathrm{VA}(M)$	languages accepted by valence automata over M.
$\mathrm{VA}^+(M)$	languages accepted by VA over M without silent transitions

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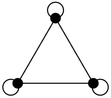
Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^* / \{a\bar{a} = 1\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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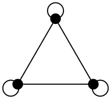

$\mathbb{Z}^3$

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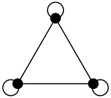

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Blind multicounter

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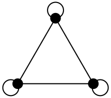

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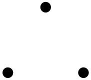

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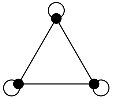

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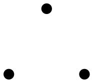

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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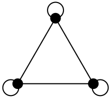

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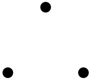

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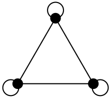

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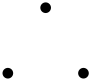

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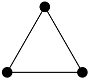

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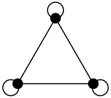

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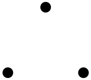

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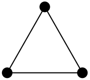

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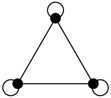

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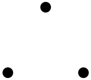

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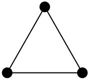

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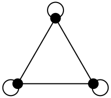

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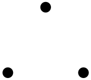

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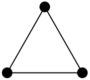

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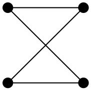

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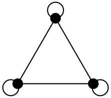

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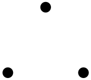

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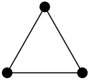

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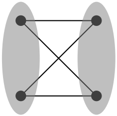

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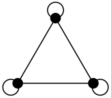

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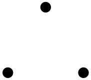

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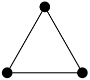

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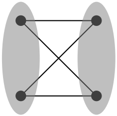

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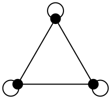

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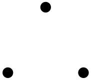

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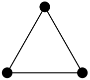

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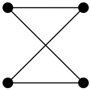

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Infinite tape (TM)

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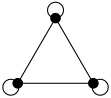

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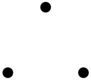

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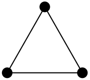

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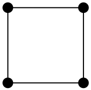

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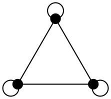

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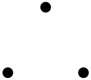

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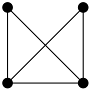

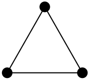

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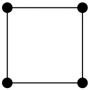
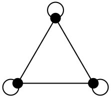


$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

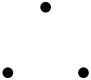
Infinite tape (TM)

Examples



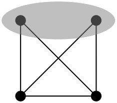
$\mathbb{Z}^3$

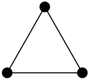
Blind multicounter



$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown





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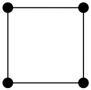

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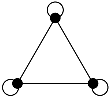

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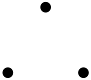

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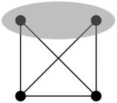

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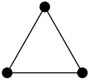

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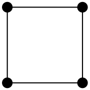

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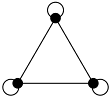

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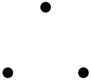

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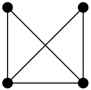

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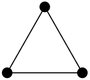
Pushdown + partially blind counters



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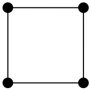

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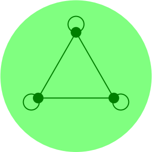

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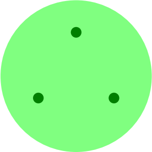

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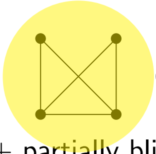

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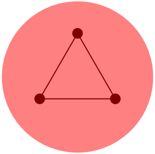

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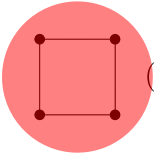

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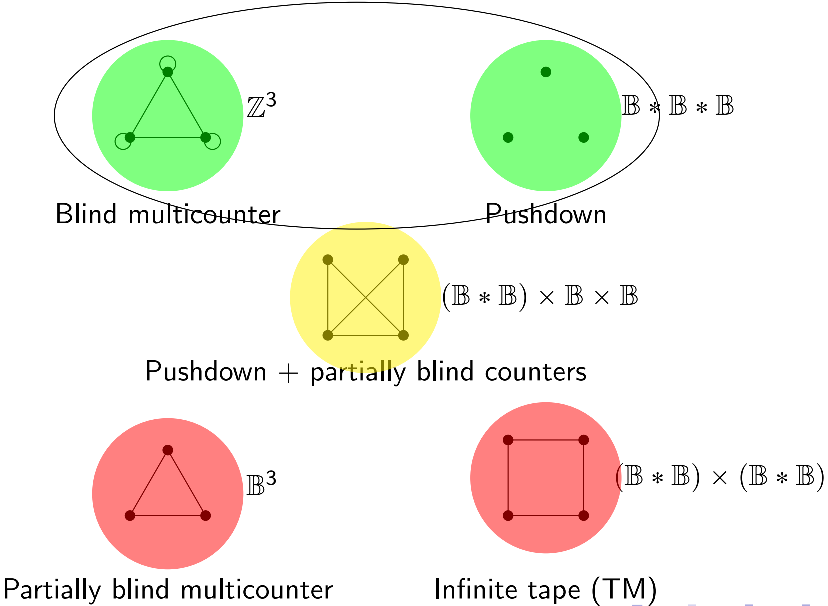

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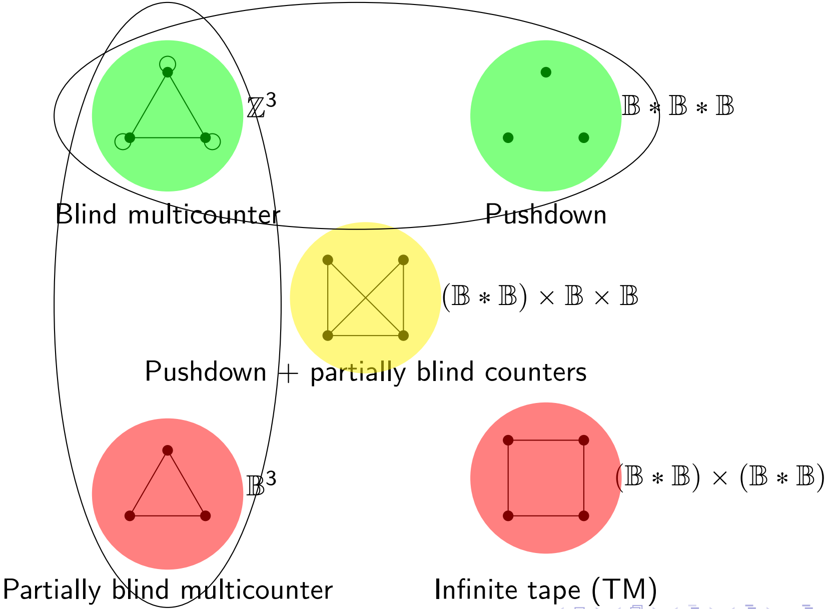
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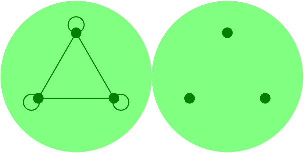
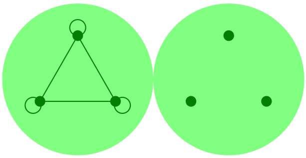
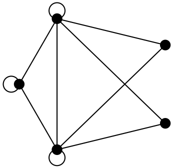


Theorem

Let Γ be a graph such that

- *any two looped vertices are adjacent*
- *no two unlooped vertices are adjacent*





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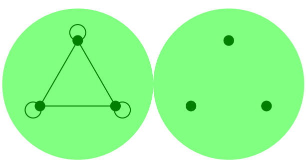

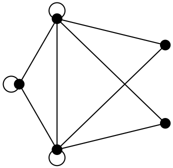

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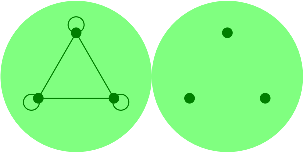
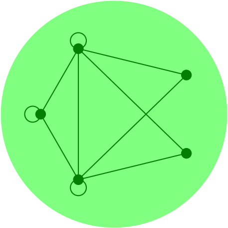
Then $\mathsf{VA}(\mathbb{M}\Gamma) = \mathsf{VA}^{+}(\mathbb{M}\Gamma)$ *if and only if* Γ *does not contain*

as an induced subgraph.





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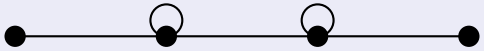

as an induced subgraph.

Negative case

By reduction to an undecidable problem from group theory (Lohrey, Steinberg 2008), we obtain:

Lemma

Let Γ *be a graph with*

as an induced subgraph. Then $\mathsf{VA}(\mathbb{M}\Gamma)$ *contains an undecidable language. Hence,* $\mathsf{VA}^+(\mathbb{M}\Gamma) \subsetneq \mathsf{VA}(\mathbb{M}\Gamma)$.

Positive case

Definition

Let $\mathcal{C}$ be the smallest class of monoids such that

- $1 \in \mathcal{C}$
- if $M \in \mathcal{C}$, then $M \times \mathbb{Z} \in \mathcal{C}$
- if $M \in \mathcal{C}$, then $M * \mathbb{B} \in \mathcal{C}$

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Lemma

Let Γ be a graph such that

- *any two looped vertices are adjacent*
- *no two unlooped vertices are adjacent*
-  *does not appear as an induced subgraph*

Then, $\mathbb{M}\Gamma \in \mathcal{C}$.

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Interpretation of $\mathcal{C}$

$\mathcal{C}$ corresponds to the class of storage mechanisms obtained by

- adding a blind counter ($M \times \mathbb{Z}$) and
- building stacks ($M * \mathbb{B}$).

λ-Elimination I

Lemma

For $M \in \mathcal{C}$, every language in $\mathrm{VA}(M)$ has semilinear Parikh image.

- $M \mapsto M \times \mathbb{Z}$, $M \mapsto M * \mathbb{B}$ preserve semilinearity

λ-Elimination I

Definition

Let $\mathcal{F}$ be a family. An *$\mathcal{F}$-grammar* is a quadruple $G = (N, T, P, S)$ where

- N, T are disjoint alphabets,
- P is a finite set of pairs $A \to M$, with $A \in N$ and $M \subseteq (N \cup T)^*$, $M \in \mathcal{F}$,
- $S \in N$.

$x \Rightarrow_G y$: if $x = uAv$ and $y = uwv$ for some $u, v, w \in (N \cup T)^*$ and $A \to M \in P$ with $w \in M$.

$$L(G) = \{w \in T^* \mid S \Rightarrow_G^* w\}.$$

L is called *algebraic over $\mathcal{F}$* if there is an $\mathcal{F}$-grammar G such that $L = L(G)$.

λ-Elimination I

Theorem (van Leeuwen 1974)

Let $\mathcal{F}$ be a family of semilinear languages. Then any language that is algebraic over $\mathcal{F}$ is also semilinear.

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Lemma

Every language in $\mathsf{VA}(M * M')$ *is algebraic over* $\mathsf{VA}(M) \cup \mathsf{VA}(M')$.

λ-Elimination II

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Definition

$\mathsf{VT}(M, C)$	Transductions $T \subseteq X^* \times C$ by valence transducers over M
$\mathsf{VT}^+(M, C)$	performed by λ-free transducers

M is called *strongly λ-independent* if

$$\mathsf{VT}(M, C) = \mathsf{VT}^+(M, C)$$

for every commutative monoid C.

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$$\mathsf{VT}(M, C) = \Phi(\mathsf{VT}^+(M, \mathsf{SL}(C)))$$

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Observation

If M is strongly λ-independent, then $\mathsf{VA}^+(M) = \mathsf{VA}(M)$.

Elimination of λ-transitions III

Definition

A subset $S \subseteq M$ is called *rational* if it is the homomorphic image of a regular language.

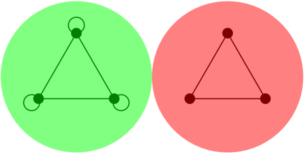
Rational subsets of $M \times C$

- For a given pair of non-λ-transitions, the set of $(m, c) \in M \times C$ applied in between is a rational set.
- Normal form for rational subsets of $M \times C$: first pop (+counter+output), then push (+counter+output)
- Modification of well-known technique for monadic rewriting systems
- Gluing in automata accepting semilinear sets

Elimination of λ-transitions IV

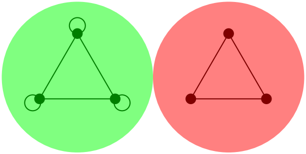
Construction for $\mathsf{VA}^+(\mathbb{M}\Gamma) = \mathsf{VA}(\mathbb{M}\Gamma)$

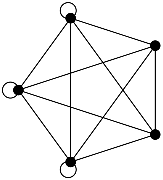
- Separate constructions for $\mathbb{B}$, $M \times \mathbb{Z}$, and $M * \mathbb{B}$.
- Transform the automaton so as to simulate the application of a rational set in one step.
- Representations of rational sets are encoded into the state or the monoid elements.
- When simulating cancellations, output semilinear sets.



Theorem

Let Γ be a graph such that between any two distinct vertices, there is an edge.





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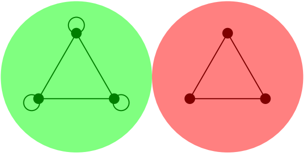

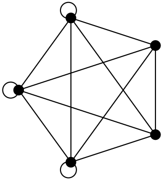

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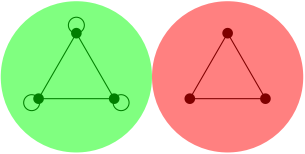
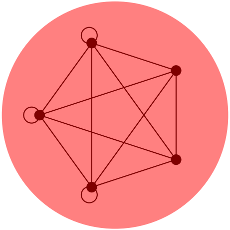
Let Γ be a graph such that between any two distinct vertices, there is an edge. Then $\mathsf{VA}(\mathbb{M}\Gamma) = \mathsf{VA}^+(\mathbb{M}\Gamma)$ *if and only if the number of unlooped nodes is* $\leqslant 1$.





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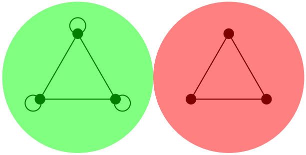

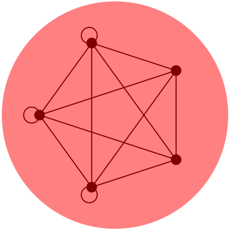

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$$\mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s) = \mathsf{VA}^{+}(\mathbb{B}^r \times \mathbb{Z}^s) \quad \textit{iff} \quad r \leqslant 1.$$

Observation

$\mathsf{VA}(\mathbb{B} \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B} \times \mathbb{Z}^s)$ already follows from the first theorem.

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Proving $\mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \subsetneq \mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r \geqslant 2$

- Use Greibach's and Jantzen's language

$$L_1 = \{wc^n \mid w \in \{0,1\}^*, n \leqslant \mathrm{bin}(w)\},$$

$$\mathrm{bin}(v0) = 2 \cdot \mathrm{bin}(v), \quad \mathrm{bin}(v1) = 2 \cdot \mathrm{bin}(v) + 1, \quad \mathrm{bin}(\lambda) = 0.$$

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- Known that $L_1 \in \mathsf{VA}(\mathbb{B}^2)$.
- Show $L_1 \notin \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r, s \in \mathbb{N}$.

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- Count *fooling sets*, concept from state complexity.

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- Use Greibach's and Jantzen's language

$$L_1 = \{wc^n \mid w \in \{0,1\}^*, n \leqslant \mathrm{bin}(w)\},$$

$$\mathrm{bin}(v0) = 2 \cdot \mathrm{bin}(v), \quad \mathrm{bin}(v1) = 2 \cdot \mathrm{bin}(v) + 1, \quad \mathrm{bin}(\lambda) = 0.$$

- Known that $L_1 \in \mathsf{VA}(\mathbb{B}^2)$.
- Show $L_1 \notin \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s)$ for $r, s \in \mathbb{N}$.
- Count *fooling sets*, concept from state complexity.
- Languages in $\mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s)$ have polynomially many fooling sets
- L_1 has exponential number of fooling sets

Thank you for your attention!