Rational subsets and submonoids of wreath products

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Rational sets in arbitrary monoids: Definition 1

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The set $\mathrm{Rat}(M) \subseteq 2^M$ of all rational subsets of M is the smallest set such that:

- Every finite subset of M belongs to $\mathrm{Rat}(M)$.
- If $L_1, L_2 \in \mathrm{Rat}(M)$, then also $L_1 \cup L_2,\ L_1 L_2 \in \mathrm{Rat}(M)$.
- If $L \in \mathrm{Rat}(M)$, then also $L^* \in \mathrm{Rat}(M)$.

Rational sets in arbitrary monoids: Definition 2

A finite automaton over M is a tuple $A = (Q, \Delta, q_0, F)$ where

- Q is a finite set of states,
- $q_0 \in Q$, $F \subseteq Q$, and
- $\Delta \subseteq Q \times M \times Q$ is finite.

The subset $L(A) \subseteq M$ is the set of all products $m_1 m_2 \cdots m_k$ such that there exist $q_1, \ldots, q_k \in Q$ with

$$(q_{i-1}, m_i, q_i) \in \Delta \text{ for } 1 \leq i \leq k \text{ and } q_k \in F.$$

Then:

$$L \in \mathrm{Rat}(M) \iff \exists \text{ finite automaton } A \text{ over } M \ : L(A) = L$$

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The rational subset membership problem for G (RatMP(G)) is the following computational problem:

INPUT: A finite automaton A over G and $g \in G$

QUESTION: $g \in L(A)$?

Membership in submonoids/subgroups

The submonoid membership problem for G is the following computational problem:

INPUT: A finite subset $A \subseteq G$ and $g \in G$

QUESTION: $g \in A^*$?

The subgroup membership problem for G (or generalized word problem for G) is the following computational problem:

INPUT: A finite subset $A \subseteq G$ and $g \in G$

QUESTION: $g \in \langle A \rangle$ $(= (A \cup A^{-1})^*)$?

The generalized word problem is a widely studied problem in combinatorial group theory.

Wreath products

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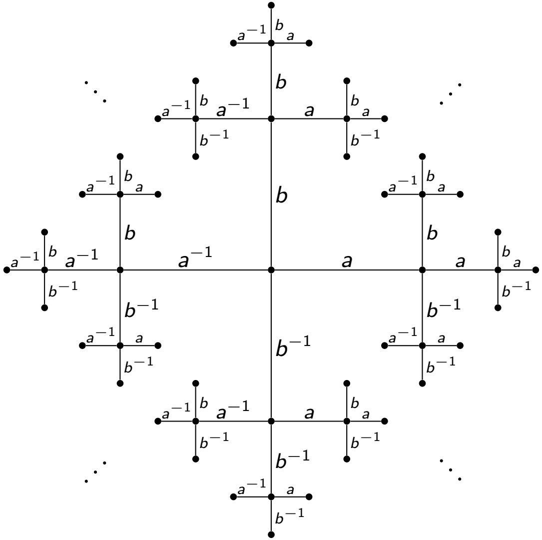
The wreath product $A \wr B$ is the set of all pairs $K \times B$ with the following multiplication, where $(k_1, b_1), (k_2, b_2) \in K \times B$:

$$(k_1, b_1)(k_2, b_2) = (k, b_1 b_2) \text{ with } \forall b \in B : k(b) = k_1(b) k_2(b_1^{-1} b).$$

Wreath product $\mathbb{Z}_2 \wr F(a,b)$ with $\mathbb{Z}_2 = \langle c \mid c^2 = 1 \rangle$

$cbcb^{-1}cabcb^{-1}ca$:



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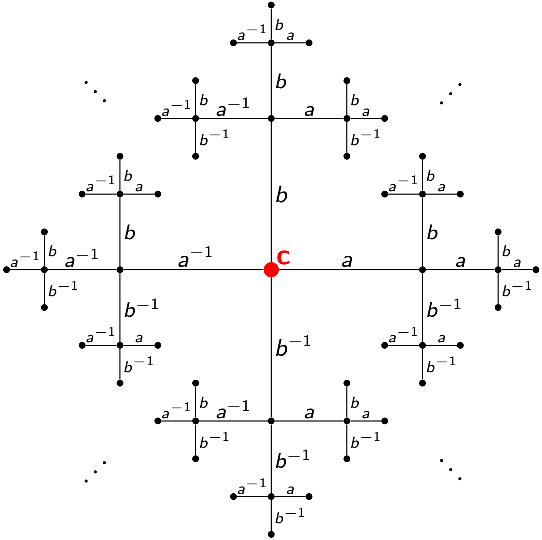

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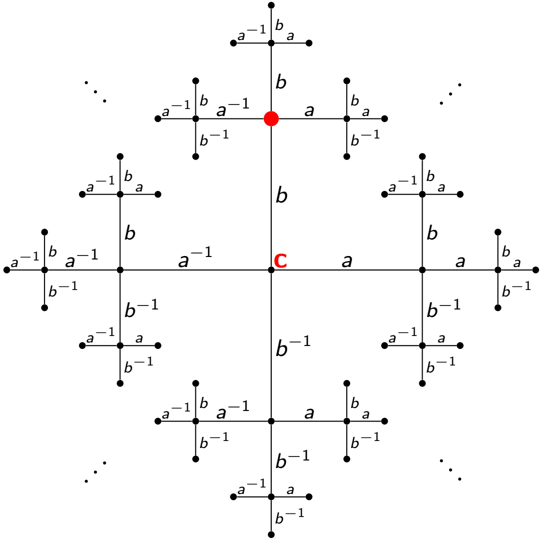

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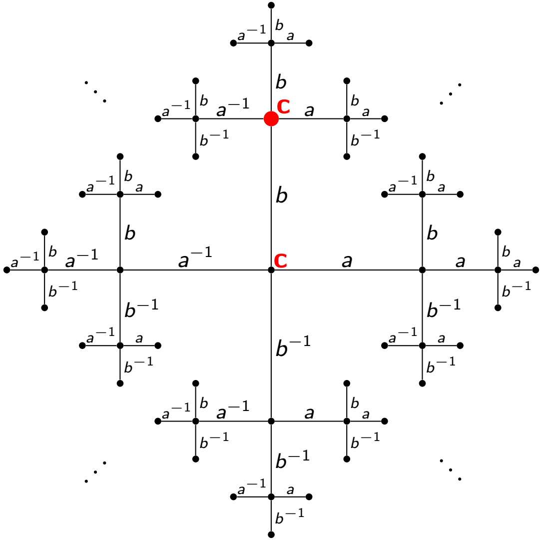

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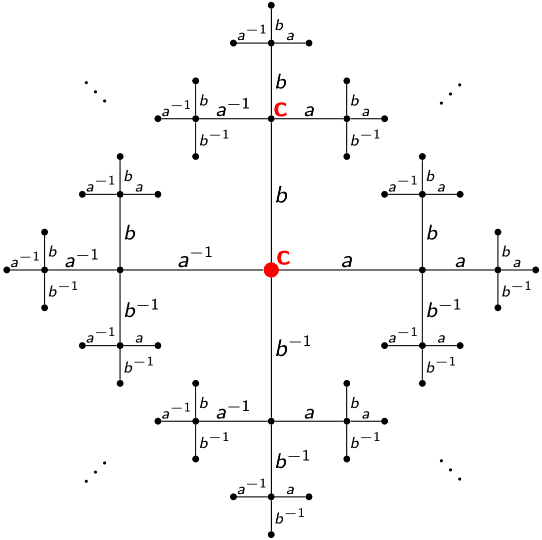

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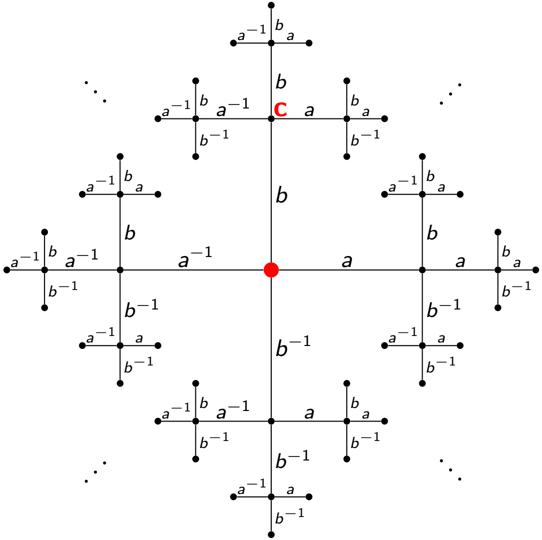

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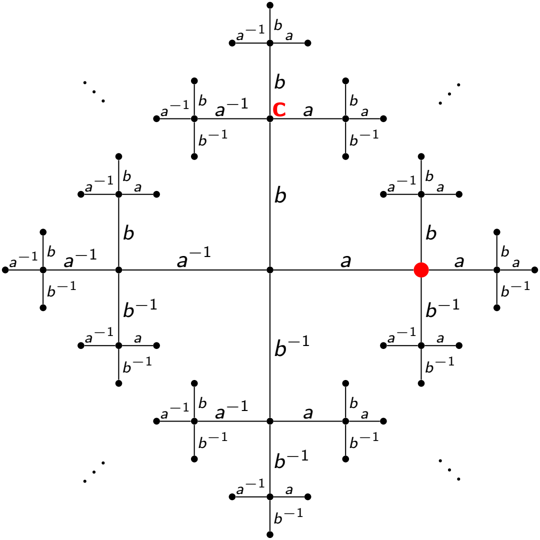

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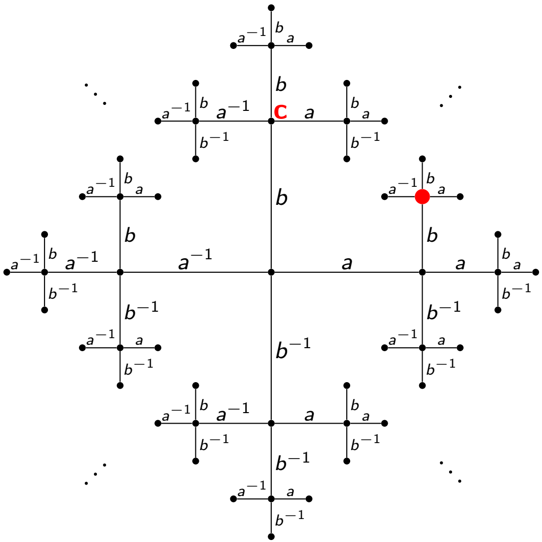

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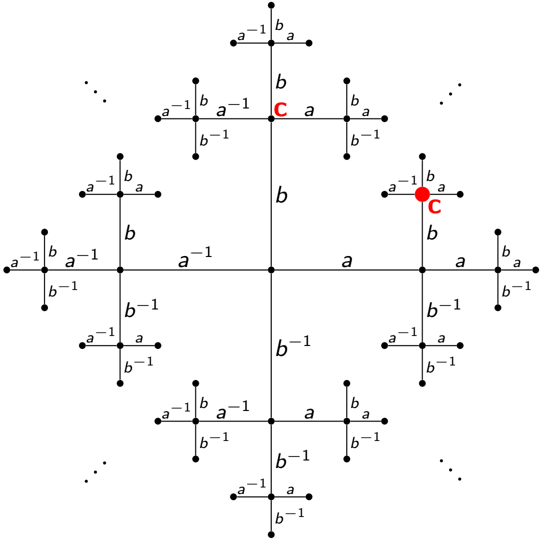

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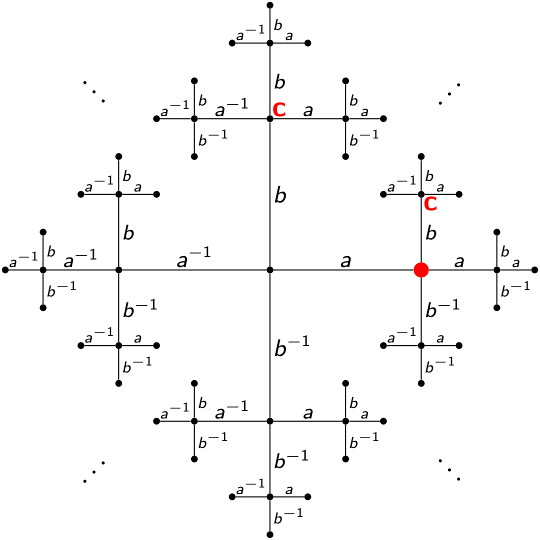

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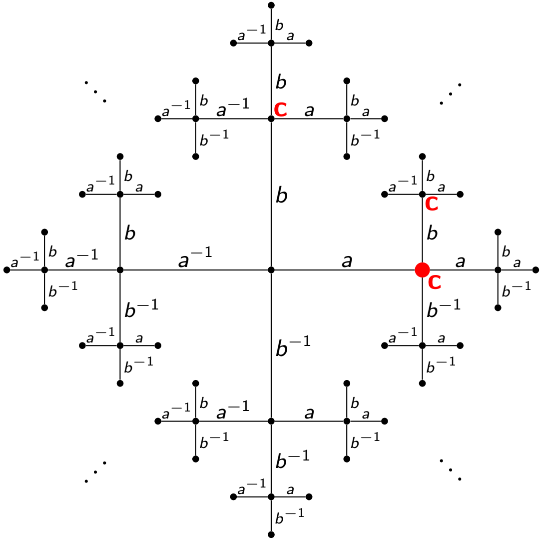

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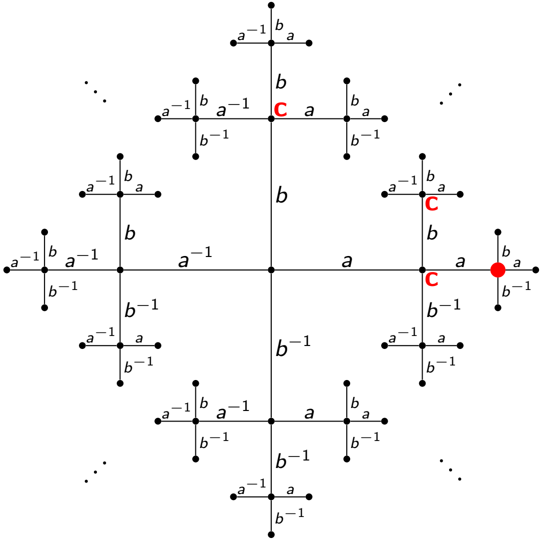

Rational subsets in wreath products: Undecidability

Lohrey, Steinberg 2009

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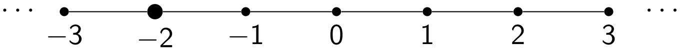

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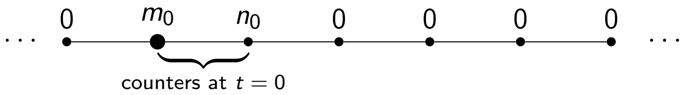

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Fix an automaton $A = (Q, \Delta, q_0, F)$ over the finite alphabet $H \cup \{a, b, a^{-1}, b^{-1}\}$.

We want to check whether there is a $w \in L(A)$ with $w = 1$ in G.

Let $p, q \in Q$, $d \in \{a, b, a^{-1}, b^{-1}\}$. A (p, d, q)-loop is an A-path

$$\pi = (p = p_0 \xrightarrow{d} p_1 \xrightarrow{\alpha_1} p_2 \xrightarrow{\alpha_2} p_3 \cdots \xrightarrow{\alpha_{n-1}} p_n \xrightarrow{d^{-1}} p_{n+1} = q)$$

with the following properties, where $\alpha_1 \cdots \alpha_i = (k_i, u_i) \in H \wr F_2$ for $1 \leq i \leq n-1$:

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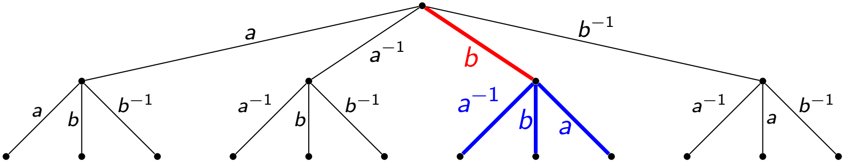
- effect$(\pi) = d\alpha_1 \cdots \alpha_{n-1} d^{-1} \in K$.

Loops

For all types $t \in \{1, a, a^{-1}, b, b^{-1}\}$ define

$$C_t \quad = \quad \{a, a^{-1}, b, b^{-1}\} \setminus \{t^{-1}\}$$



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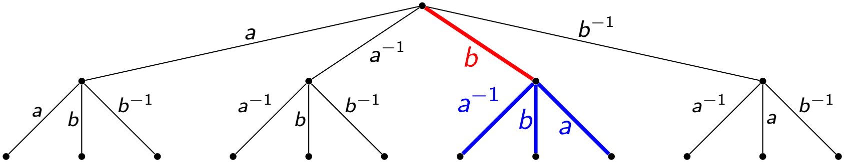

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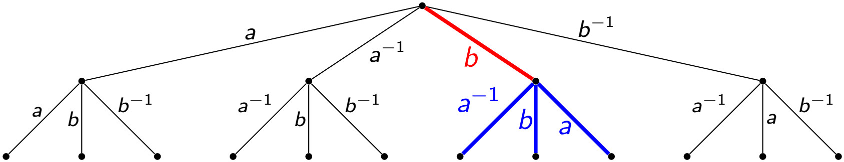

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Observation

The alphabet X_t can be computed.

Loop patterns

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A loop pattern at t is a word

$$w = (p_1, d_1, q_1)(p_2, d_2, q_2) \cdots (p_n, d_n, q_n) \in X_t^*.$$

such that for every $1 \leq i \leq n$ there exists a (p_i, d_i, q_i)-loop π_i with

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- P_t is regular and
- an automaton for P_t can be computed.

A well quasi order

A WQO (well quasi order) is a reflexive and transitive relation $\preceq$ (on a set A) such that for every infinite sequence $a_1, a_2, a_3, \ldots$ there exist $i < j$ with $a_i \preceq a_j$.

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For a group H, we define a partial order $\preceq_H$ on X^* (X any finite alphabet) as follows: $u \preceq_H v$ iff there exist factorizations

$$\begin{aligned} u &= x_1 x_2 \cdots x_n \quad (x_i \in X) \\ v &= v_0 x_1 v_1 x_2 \cdots v_{n-1} x_n v_n \end{aligned}$$

such that for every homomorphism $\varphi : X^* \to H$ we have $\varphi(v_0) = \varphi(v_1) = \cdots \varphi(v_n) = 1$.

A well quasi order

A WQO (well quasi order) is a reflexive and transitive relation $\preceq$ (on a set A) such that for every infinite sequence $a_1, a_2, a_3, \ldots$ there exist $i < j$ with $a_i \preceq a_j$.

For a group H, we define a partial order $\preceq_H$ on X^* (X any finite alphabet) as follows: $u \preceq_H v$ iff there exist factorizations

$$\begin{aligned} u &= x_1 x_2 \cdots x_n \quad (x_i \in X) \\ v &= v_0 x_1 v_1 x_2 \cdots v_{n-1} x_n v_n \end{aligned}$$

such that for every homomorphism $\varphi : X^* \to H$ we have $\varphi(v_0) = \varphi(v_1) = \cdots \varphi(v_n) = 1$.

Lemma

For every finite group H, $\preceq_H$ is a WQO.

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For every $t \in \{1, a, a^{-1}, b, b^{-1}\}$, *the set of loop patterns* P_t *is upward closed w.r.t.* $\preceq_H$.

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This implies that P_t is regular, but can we compute an NFA for P_t?

A fixpoint characterization of P_t

For $i \in \mathbb{N}$, let $P_t^{(i)} \subseteq X_t^*$ be the set of loop patterns of depth $\leq i$.

Observation

There is an operator Φ with

$$\Phi\left[(P_t^{(i)})_{t\in T}\right] = (P_t^{(i+1)})_{t\in T}$$

such that Φ is effectively regularity perserving: For regular sets R_t, the languages in the tuple $\Phi\left[(R_t)_{t\in T}\right]$ are effectively regular.

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Lemma

$(P_t)_{t \in T}$ is the smallest fixpoint of Φ containing $(\{\lambda\})_{t \in T}$.

Algorithm

1: $U_t^{(0)} := \{\lambda\}{\uparrow_H}$ for each $t \in T$.
2: **while** $\exists w \in \Phi\left[(U_t^{(i)})_{t\in T}\right]_t \setminus U_t^{(i)}$ for some $t \in T$ **do**
3: $\quad U_t^{(i+1)} := U_t^{(i)} \cup \{w\}{\uparrow_H}$
4: $\quad i := i + 1$
5: **end while**

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- Thus $U_t^{(k)} = P_t$.

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- Conjecture: Whenever H is non-trivial and G is not virtually-free, then RatMP($H \wr G$) is undecidable.
- Is there a (necessarily one-ended) group G, for which the submonoid membership problem is decidable but RatMP(G) is undecidable?