The monoid of queue actions

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Abstract. We model the behavior of a fifo-queue as a monoid of transformations that are induced by sequences of writing and reading. We describe this monoid by means of a confluent and terminating semi-Thue system and study some of its basic algebraic properties such as conjugacy. Moreover, we show that while several properties concerning its rational subsets are undecidable, their uniform membership problem is NL-complete. Furthermore, we present an algebraic characterization of this monoid's recognizable subsets. Finally, we prove that it is not Thurston-automatic.

1 Introduction

Basic computing models differ in their storage mechanisms: there are finite memory mechanisms, counters, blind counters, partially blind counters, pushdowns, Turing tapes, queues and combinations of these mechanisms. Every storage mechanism naturally comes with a set of basic actions like reading a symbol from or writing a symbol to the pushdown. As a result, sequences of basic actions transform the storage. The set of transformations induced by sequences of basic actions then forms a monoid. As a consequence, fundamental properties of a storage mechanism are mirrored by algebraic properties of the induced monoid. For example, the monoid induced by a deterministic finite automaton is finite, a single blind counter induces the integers with addition, and pushdowns induce polycyclic monoids [10]. In this paper, we are interested in a queue as a storage mechanism. In particular, we investigate the monoid $\mathcal{Q}$ induced by a single queue.

The basic actions on a queue are writing the symbol a into the queue and reading the symbol a from the queue (for each symbol a from the alphabet of the queue). Since a can only be read from a queue if it is the first entry in the queue, these actions are partial. Hence, for every sequence of basic actions, there is a queue of shortest length that can be transformed by the sequence without error (i.e., without attempting to read a from a queue that does not start with a). Our first main result (Theorem 4.1) in section 4 provides us with a normal form for transformations induced by sequences of basic actions: The transformation induced by a sequence of basic actions is uniquely determined by the subsequence of write actions, the subsequence of read actions, and the length of the shortest queue that can be transformed by the sequence without error. The proof is based on a convergent finite semi-Thue system for the monoid $\mathcal{Q}$. In sections 3 and 5, we derive equations that hold in $\mathcal{Q}$. The main result in this direction is Theorem 5.3, which describes the normal form of the product of two sequences of basic actions in normal form, i.e., it describes the monoid operation in terms of normal forms.

Sections 6 and 7 concentrate on the conjugacy problem in $\mathcal{Q}$. The fundamental notion of conjugacy in groups has been extended to monoids in two different ways: call x and y conjugate if the equation $xz = zy$ has a solution, and call them transposed if there are u and v such that $x = uv$ and $y = vu$. Then conjugacy $\approx$ is reflexive and transitive, but not necessarily symmetric, and transposition $\sim$ is reflexive and symmetric, but not necessarily transitive. These two relations have been considered, e.g., in [13, 16, 17, 6, 18, 5]. We prove that conjugacy is the transitive closure of transposition and that two elements of $\mathcal{Q}$ are conjugate if and only if their subsequences of write and of read actions, respectively, are conjugate in the free monoid. This characterization allows in particular to decide conjugacy in polynomial time. In section 7, we prove that the set of solutions $z \in \mathcal{Q}$ of $xz = zy$ is effectively rational but not necessarily recognizable.

Section 8 investigates algorithmic properties of rational subsets of $\mathcal{Q}$. Algorithmic aspects of rational subsets have received increased attention in recent years; see [14] for a survey on the membership problem. Employing the fact that every element of $\mathcal{Q}$ has only polynomially many left factors, we can nondeterministically solve the rational subset membership problem in logarithmic space. Since the direct product of two free monoids embeds into $\mathcal{Q}$, all the negative results on rational transductions (cf. [1]) as, e.g., the undecidability of universality of a rational subset, translate into our setting (cf. Theorem 8.3). The subsequent section 9 characterizes the recognizable subsets of $\mathcal{Q}$. Recall that an element of $\mathcal{Q}$ is completely determined by its subsequences of write and read actions, respectively, and the length of the shortest queue that can be transformed without an error. Regular conditions on the subsequences of write and read actions, respectively, lead to recognizable sets in $\mathcal{Q}$. Regarding the shortest queue that can be transformed without error, the situation is more complicated: the set of elements of $\mathcal{Q}$ that operate error-free on the empty queue is not recognizable. Using an approximation of the length of the shortest queue, we obtain recognizable subsets $\Omega_k \subseteq \mathcal{Q}$. The announced characterization then states that a subset of $\mathcal{Q}$ is recognizable if and only if it is a Boolean combination of regular conditions on the subsequences of write and read actions, respectively, and sets Ω_k (cf. Theorem 9.4). In the final section 10, we prove that $\mathcal{Q}$ is not automatic in the sense of Thurston et al. [4] (it cannot be automatic in the sense of Khoussainov and Nerode [12] since the free monoid with two generators is interpretable in first order logic in $\mathcal{Q}$).

All missing proofs are contained in the complete version [9] of this paper.

2 Preliminaries

Let A be an alphabet. As usual, the set of finite words over A, i.e. the free monoid generated by A, is denoted A^*. Let $w = a_1 \dots a_n \in A^*$ be some word. The *length* of w is $|w| = n$. The word obtained from w by reversing the order of its symbols is $w^R = a_n \dots a_1$. A word $u \in A^*$ is a *prefix* of w if there is $v \in A^*$ such that $w = uv$. In this situation, the word v is unique and we refer to it by $u^{-1}w$. Similarly, u is a *suffix* of w if $w = vu$ for some $v \in A^*$ and we then put $wu^{-1} = v$. For $k \in \mathbb{N}$, we let $A^{\leq k} = \{\, w \in A^* \mid |w| \leq k \,\}$ and define $A^{>k}$ similarly.

Let M be an arbitrary monoid. The *concatenation* of two subsets $X, Y \subseteq M$ is defined as $X \cdot Y = \{\, xy \mid x \in X, y \in Y \,\}$. The *Kleene iteration* of X is the set

$X^* = \{ x_1 \cdots x_n \mid n \in \mathbb{N}, x_1, \ldots, x_n \in X \}$. In fact, X^* is a submonoid of M, namely the smallest submonoid entirely including X. Thus, X^* is also called *the submonoid generated by* X. The monoid M is *finitely generated*, if there is some finite subset $X \subseteq M$ such that $M = X^*$.

A subset $L \subseteq M$ is called *rational* if it can be constructed from the finite subsets of M using union, concatenation, and Kleene iteration only. The subset L is *recognizable* if there are a finite monoid F and a morphism $\phi\colon M \to F$ such that $\phi^{-1}(\phi(L)) = L$. The image of a rational set under a monoid morphism is again rational, whereas recognizability is retained under preimages of morphisms. It is well-known that every recognizable subset of a finitely generated monoid is rational. The converse implication is in general false. However, if M is the free monoid generated by some alphabet A, a subset $L \subseteq A^*$ is rational if and only if it is recognizable. In this situation, we call L *regular*.

3 Definition and basic equations

We want to model the behavior of a fifo-queue whose entries come from a finite set A with $|A| \geq 2$ (if A is a singleton, the queue degenerates into a partially blind counter). Consequently, the state of a valid queue is an element from A^*. In order to have a defined result even if a read action fails, we add the error state $\bot$. The basic actions are writing of the symbol $a \in A$ into the queue (denoted a) and reading the symbol $a \in A$ from the queue (denoted $\overline{a}$). Formally, $\overline{A}$ is a disjoint copy of A whose elements are denoted $\overline{a}$. Furthermore, we set $\Sigma = A \cup \overline{A}$. Hence, the free monoid Σ^* is the set of sequences of basic actions and it acts on the set $A^* \cup \{\bot\}$ by way of the function $.\colon (A^* \cup \{\bot\}) \times \Sigma^* \to A^* \cup \{\bot\}$, which is defined as follows:

$$q.\varepsilon = q \qquad q.au = qa.u \qquad q.\overline{a}u = \begin{cases} q'.u & \text{if } q = aq' \\ \bot & \text{otherwise} \end{cases} \qquad \bot.u = \bot$$

for $q \in A^*$, $a \in A$, and $u \in \Sigma^*$.

Example 3.1. Let the content of the queue be $q = ab$. Then $ab.\overline{a}c = b.c = bc.\varepsilon = bc$ and $ab.c\overline{a} = abc.\overline{a} = bc.\varepsilon = bc$, i.e., the sequences of basic actions $\overline{a}c$ and $c\overline{a}$ behave the same on the queue $q = ab$. In Lemma 3.5, we will see that this is the case for any queue $q \in A^* \cup \{\bot\}$. Differently, we have $\varepsilon.\overline{a}a = \bot \neq \varepsilon = \varepsilon.a\overline{a}$, i.e., the sequences of basic actions $a\overline{a}$ and $\overline{a}a$ behave differently on certain queues.

Definition 3.2. *Two words $u, v \in \Sigma^*$ are* equivalent *if $q.u = q.v$ for all queues $q \in A^*$. In that case, we write $u \equiv v$. The equivalence class wrt. $\equiv$ containing the word u is denoted $[u]$.*

Since $\equiv$ is a congruence on the free monoid Σ^*, we can define the quotient monoid $\mathcal{Q} = \Sigma^*/{\equiv}$ and the natural epimorphism $\eta\colon \Sigma^* \to \mathcal{Q}\colon w \mapsto [w]$. The monoid $\mathcal{Q}$ is called the monoid of queue actions.

Remark 3.3. Note that the concrete form of $\mathcal{Q}$ depends on the size of the alphabet A, so let $\mathcal{Q}_n$ denote the monoid of queue actions defined with $A = |n|$. As a consequence of Theorems 4.1 and 5.3 below, $\mathcal{Q}_n$ embeds into $\mathcal{Q}_2$ where the generators of $\mathcal{Q}_n$ are mapped to $[a^{n+i}ba^{n-i}b]$ and $[\overline{a}^{n+i}\overline{b}\overline{a}^{n-i}\overline{b}]$, respectively.

Intuitively, the basic actions a and $\overline{a}$ act "dually" on $A^* \cup \{\bot\}$. We formalize this intuition by means of the *duality map* $\delta\colon \Sigma^* \to \Sigma^*$, which is defined as follows: $\delta(\varepsilon) = \varepsilon$, $\delta(au) = \delta(u)\overline{a}$, and $\delta(\overline{a}u) = \delta(u)a$ for $a \in A$ and $u \in \Sigma^*$. Notice that $\delta(uv) = \delta(v)\delta(u)$ and $\delta(\delta(u)) = u$ (i.e., δ is an anti-isomorphism and an involution). In the following, we use the term "by duality" to refer to the proposition below.

Proposition 3.4. *For $u, v \in \Sigma^*$, we have $u \equiv v$ if and only if $\delta(u) \equiv \delta(v)$.*

Consequently, the duality map δ can be lifted to a map $\delta\colon \mathcal{Q} \to \mathcal{Q}\colon [u] \mapsto [\delta(u)]$. Also this lifted map is an anti-isomorphism of $\mathcal{Q}$ and an involution.

The second equivalence in the lemma below follows from the first one by duality.

Lemma 3.5. *Let $a, b \in A$. We have $ab\overline{b} \equiv a\overline{b}b$, $a\overline{a}\overline{b} \equiv \overline{a}a\overline{b}$, and if $a \neq b$ then $a\overline{b} \equiv \overline{b}a$.*

From the first and the last equivalence, we get $ab\overline{c} \equiv a\overline{c}b$ for any $a, b, c \in A$, even when $b = c$. Similarly, the second and the third equivalence imply $a\overline{b}\overline{c} \equiv \overline{b}a\overline{c}$.

Our computations in $\mathcal{Q}$ will frequently make use of alternating sequences of write- and read-operations on the queue. To simplify notation, we define the shuffle of two words over A and over $\overline{A}$ as follows: Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$ with $v = a_1 a_2 \ldots a_n$ and $w = b_1 b_2 \ldots b_n$. We write $\overline{w}$ for $\overline{b_1}\,\overline{b_2}\,\ldots\,\overline{b_n}$ and set

$$\langle v, \overline{w} \rangle = a_1\overline{b_1}\,a_2\overline{b_2}\,\ldots\,a_n\overline{b_n}\,.$$

The following proposition describes the relation between the shuffle operation and the multiplication in $\mathcal{Q}$. Its proof works by induction on the lengths of x and y.

Proposition 3.6. *Let $u, v, x, y, x', y' \in A^*$.*

(1) if $xy = x'y'$ and $|x| = |y'| = |u|$, then $\langle u, \overline{x} \rangle\, \overline{y} \equiv \overline{x'}\, \langle u, \overline{y'} \rangle$.
(2) if $xy = x'y'$ and $|y| = |x'| = |v|$, then $x\, \langle y, \overline{v} \rangle \equiv \langle x', \overline{v} \rangle\, y'$.
(3) If $|u| = |v|$ and $|x| = |y|$, then $x\, \langle u, \overline{v} \rangle\, \overline{y} \equiv \langle xu, \overline{vy} \rangle$.
(4) If $|x| = |y|$, then $\langle x, \overline{y} \rangle \equiv x\overline{y}$.

The first claim expresses that the sequence of write-operations u can be "moved along" the sequence of read-operations $\overline{xy} = \overline{x'y'}$, its dual (2) moves a sequence of read-operations $\overline{v}$ along a sequence of write-operations. The third claim expresses that write-operations from the left and read-operations from the right can be "swallowed" by a shuffle. The last one follows from (3) with $u = v = \varepsilon$.

Corollary 3.7. *Let $u, v, w \in A^*$. If $|w| = |v|$, then $\overline{u}v\overline{w} \equiv v\overline{uw}$. If $|u| = |v|$, then $u\overline{v}w \equiv uw\overline{v}$.*

The first claim follows from $v\overline{w} \equiv \langle v, \overline{w} \rangle$ and the possibility to move v along the sequence of read-operations $\overline{uw}$, the second claim follows dually.

4 A semi-Thue system for $\mathcal{Q}$

We order the equations from Lemma 3.5 as follows (with $a \neq c$):

$$ab\overline{b} \to a\overline{b}b \qquad a\overline{a}\overline{b} \to \overline{a}a\overline{b} \qquad a\overline{c} \to \overline{c}a$$

Let R be the semi-Thue system with the above three types of rules. Note that a word over Σ is irreducible if and only if it has the form $\overline{u}\,\langle v, \overline{v}\rangle\, w$ for some $u, v, w \in A^*$. When doing our calculations, we found it convenient to think in terms of pictures as follows:

$\overline{u}$	$\overline{v}$	
	v	w

Here, the blocks represent the words $\overline{u}$, $\overline{v}$, v, and w, respectively, where we placed the read-blocks (i.e., words over $\overline{A}$) in the first line and write-blocks in the second. The shuffle $\langle v, \overline{v}\rangle$ is illustrated by placing the corresponding two blocks on top of each other.

Ordering the alphabet such that $\overline{a} < b$ for all $a, b \in A$, all rules are decreasing in the length-lexicographic order; hence R is terminating. It is confluent since the only overlap of left-hand sides have the form $ab\overline{b}\overline{c}$. Consequently, for any $u \in \Sigma^*$, there is a unique irreducible word $\mathsf{nf}(u)$ with $u \xrightarrow{*} \mathsf{nf}(u)$. We call $\mathsf{nf}(u)$ the *normal form* of u and denote the set of all normal forms by $\mathsf{NF} \subseteq \Sigma^*$, i.e.,

$$\mathsf{NF} = \{\ \mathsf{nf}(u) \mid u \in \Sigma^*\ \} = \overline{A}^*\, \{\, a\overline{a} \mid a \in A\,\}^*\, A^*\,.$$

By our construction of R from the equations in Lemma 3.5, $\mathsf{nf}(u) = \mathsf{nf}(v)$ implies $u \equiv v$ for any words $u, v \in \Sigma^*$. For the converse implication, let $u \equiv v$. Because of $u \equiv \mathsf{nf}(u)$, we can assume that u and v are in normal form, i.e., $u = \overline{u_1}\,\langle u_2, \overline{u_2}\rangle\, u_3$ and $v = \overline{v_1}\,\langle v_2, \overline{v_2}\rangle\, v_3$. Then one first shows $u_1 = v_1$ using $q.u = q.v$ for $q \in \{u_1, v_1\}$. The equation $u_2 = v_2$ follows from $u_1 = v_1$ and $q.u = q.v$ for $q \in \{u_1u_2a, v_1v_2a\}$ for each $a \in A$ (here we rely on the fact that $|A| \geq 2$). Finally, $u_3 = v_3$ follows from $u_1 = v_1$, $u_2 = v_2$, and $u_1.u = u_1.v$.

Consequently, $u \equiv v$ and $\mathsf{nf}(u) = \mathsf{nf}(v)$ are equivalent. Hence, the mapping $\mathsf{nf}\colon \Sigma^* \to \mathsf{NF}$ can be lifted to a mapping $\mathsf{nf}\colon \mathcal{Q} \to \mathsf{NF}$ by defining $\mathsf{nf}([u]) = \mathsf{nf}(u)$. In summary, we have the following theorem.

Theorem 4.1. *The natural epimorphism $\eta\colon \Sigma^* \to \mathcal{Q}$ maps the set* NF *bijectively onto $\mathcal{Q}$. The inverse of this bijection is the map* $\mathsf{nf}\colon \mathcal{Q} \to \mathsf{NF}$.

Let $\pi, \overline{\pi}\colon \Sigma^* \to A^*$ be the morphisms defined by $\pi(a) = \overline{\pi}(\overline{a}) = a$ and $\pi(\overline{a}) = \overline{\pi}(a) = \varepsilon$ for $a \in A$ (i.e., π is the projection of a word over Σ to its subword over A, and $\overline{\pi}$ is the projection to its subword over $\overline{A}$, with all the bars $^-$ deleted). By Theorem 4.1, these two morphisms can be lifted to morphisms $\pi, \overline{\pi}\colon \mathcal{Q} \to A^*$ by $\pi([u]) = \pi(u)$ and $\overline{\pi}([u]) = \overline{\pi}(u)$.

Definition 4.2. *Let $w \in \Sigma^*$ be a word and $\mathsf{nf}(w) = \overline{x}\,\langle y, \overline{y}\rangle\, z$ its normal form. The* overlap width *of w and of $[w]$ is the number $\mathsf{ow}(w) = \mathsf{ow}([w]) = |y|$.*

By Theorem 4.1, $q \in \mathcal{Q}$ is uniquely determined by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$. Let $\mathsf{nf}(q) = \overline{x}\,\langle y, \overline{y}\rangle\, y$. Then x is the shortest queue w with $w.q \neq \perp$. Furthermore, $\mathsf{ow}(q) = |\overline{\pi}(q)| - |x|$. Hence, q is also uniquely described by the two projections and the length of the shortest queue it transforms without error.

5 Multiplication

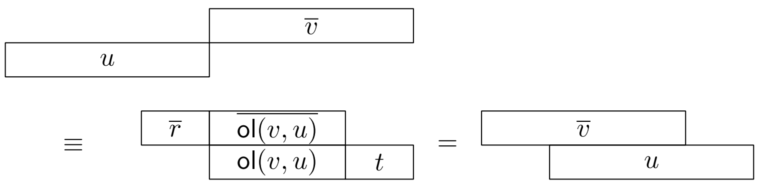
For two words u and v in normal form, we want to determine the normal form of uv. For this, the concept of *overlap* of two words will be important:

Definition 5.1. *For $u, v \in A^*$, let $\mathsf{ol}(v, u)$ denote the longest suffix of v that is also a prefix of u.*

For example, $\mathsf{ol}(ab, bc) = b$, $\mathsf{ol}(aba, aba) = aba$, and $\mathsf{ol}(ab, cba) = \varepsilon$. The following lemma describes the normal form of a word from $A^*\overline{A}^*$.

Lemma 5.2. *Let $u, v \in A^*$ and set $s = \mathsf{ol}(v, u)$, $r = vs^{-1}$ and $t = s^{-1}u$. Then $u\overline{v} \equiv \overline{r}\, \langle s, \overline{s}\rangle\, t$.*

The equation $u\overline{v} \equiv \overline{r}\, \langle s, \overline{s}\rangle\, t$ can be visualized as follows:



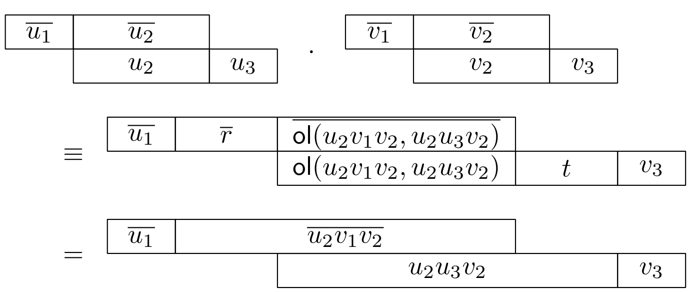
Our intuition is that the word $\overline{v}$ tries to slide along u to the left as far as possible. This movement is stopped as soon as we reach a word in normal form (which, for the first time, occurs when a suffix of v coincides with a prefix of u).

The proof of Lemma 5.2 first assumes $|u| = |v|$ and proceeds by induction on this length. The general case follows using Cor. 3.7. Applying Prop. 3.6(4), Cor. 3.7, and Lemma 5.2, one gets rather immediately the following description of the normal form of the product of two words in normal form.

Theorem 5.3. *Let $u_1, u_2, u_3, v_1, v_2, v_3 \in A^*$ and set $s = \mathsf{ol}(u_2v_1v_2, u_2u_3v_2)$, $r = u_2v_1v_2s^{-1}$, and $t = s^{-1}u_2u_3v_2$. Then*

$$\overline{u_1}\, \langle u_2, \overline{u_2}\rangle\, u_3 \cdot \overline{v_1}\, \langle v_2, \overline{v_2}\rangle\, v_3 \equiv \overline{u_1 r}\, \langle s, \overline{s}\rangle\, tv_3\,.$$

This theorem can be visualized as follows:



Here, first, v_2 moves to the left until it reaches the right border of u_3. Then $\overline{u_2}$ moves to the right until it reaches the left border of $\overline{v_1}$. Finally, the united block $\overline{u_2v_1v_2}$ tries to slide to the left along $u_2u_3v_2$ until a normal form is reached.

6 Conjugacy

The conjugacy relation in groups has two natural generalizations to monoids, which, when considered in $\mathcal{Q}$, we determine in this section. Let M be a monoid and $p, q \in M$. Then p and q are *conjugate*, which we denote by $p \approx q$, if there exists $x \in M$ such that $px = xq$. Furthermore, p and q are *transposed*, denoted by $p \sim q$, if there are $x, y \in M$ with $p = xy$ and $q = yx$. Moreover, $\overset{*}{\sim}$ is the transitive closure of $\sim$.

Observe that $\approx$ is reflexive and transitive whereas $\sim$ is reflexive and symmetric, and $\sim \subseteq \approx$. If M is actually a group, then both relations coincide and are equivalence relations, called conjugacy. The same is true for free monoids [15, Prop. 1.3.4] and even for special monoids [18], but there are monoids where none of this holds.

Example 6.1. Let $u, v, w \in A^*$. Then $\overline{u} \langle v, \overline{v} \rangle w \equiv \overline{u} v \overline{v} w \equiv v \overline{u} \overline{v} w$. Consequently, $\mathcal{Q} = \eta(A^* \overline{A}^* A^*)$ and dually $\mathcal{Q} = \eta(\overline{A}^* A^* \overline{A}^*)$. Furthermore, $v \overline{u} \overline{v} w \overset{*}{\sim} \overline{u} \overline{v} w v$. Hence, for every $q \in \mathcal{Q}$, there exists $q' \in \eta(\overline{A}^* A^*)$ with $q \overset{*}{\sim} q'$, i.e., $\mathcal{Q}$ is the closure of $\eta(\overline{A}^* A^*)$ under transposition.

Theorem 6.2. *For any $p, q \in \mathcal{Q}$, the following are equivalent:*

$$(1)\ p \overset{*}{\sim} q \qquad (2)\ p \approx q \qquad (3)\ q \approx p \qquad (4)\ (\pi(p) \sim \pi(q) \textit{ and } \overline{\pi}(p) \sim \overline{\pi}(q))$$

The implication $(1) \Rightarrow (2)$ holds in every monoid since $\approx$ is transitive and since $\sim \subseteq \approx$. The implication $(2) \Rightarrow (4)$ holds since π and $\overline{\pi}$ are homomorphisms and since $\approx$, $\sim$, and $\overset{*}{\sim}$ coincide on the free monoid. To show $(4) \Rightarrow (1)$, we first invoke Example 6.1: we can assume $p = [\overline{\pi}(p)\pi(p)]$ and similarly for q. The crucial step is to show $[\overline{x}ay] \overset{*}{\sim} [\overline{x}ya]$ and $[\overline{a}\overline{x}y] \overset{*}{\sim} [\overline{x}\overline{a}y]$, i.e., that one can rotate a single letter in the read-part or in the write-part. This ensures the implication $(4) \Rightarrow (1)$. Since (2) and (4) are equivalent and since $\sim$ is symmetric on the free monoid, also the equivalence of (2) and (3) follows.

We obtain the following consequence of Theorem 6.2: Given two words $u, v \in \Sigma^*$, one can decide in quadratic time whether $\pi(u) \sim \pi(v)$ and $\overline{\pi}(u) \sim \overline{\pi}(v)$. Consequently, it is decidable in polynomial time whether $[u] \approx [v]$ holds. It remains an open question whether there is some number $k \in \mathbb{N}$ such that $p \overset{*}{\sim} q$ if and only if $p \overset{k}{\sim} q$.

7 Conjugators

Let M be a monoid and $x, y \in M$. An element $z \in M$ is a *conjugator of x and y* if $xz = zy$. The set of all conjugators of x and y is denoted $C(x, y) = \{ z \in M \mid xz = zy \}$.

Suppose that M is a free monoid A^* and consider $x, y \in A^*$. It is well-known that $C(x, y)$ is a finite union of sets of the form $u(vu)^*$ and hence regular. In contrast, the set of conjugators of $[\overline{a}]$ and $[\overline{a}]$ is not recognizable since $\eta^{-1}(C([\overline{a}], [\overline{a}])) \cap a^* \, \overline{a}^* = \{ a^k \overline{a}^\ell \mid k \leq \ell \}$. In this section, we prove the following weaker result:

Theorem 7.1. *Let $x, y \in \mathcal{Q}$. Then the set $C(x, y)$ is rational.*

The proof proceeds as follows: First note that, by Theorem 4.1, $xz = zy$ if and only if $\pi(xz) = \pi(zy)$, $\overline{\pi}(xz) = \overline{\pi}(zy)$, and $\mathsf{ow}(xz) = \mathsf{ow}(zy)$. The set $D(x, y)$ of all $z \in \mathcal{Q}$ satisfying the first two conditions is recognizable (since x and y are fixed) and it remains to handle the third condition (under the assumption that the first two hold).

The crucial point in the proof of Theorem 7.1 is the regularity of the language

$$G_k = \{\, \mathsf{nf}(z) \mid z \in D(x,y), \mathsf{ow}(xz) - \mathsf{ow}(z) \geq k \,\} \ .$$

Having this, it follows that the languages

$$\begin{aligned} E_k &= \{\, \mathsf{nf}(z) \mid z \in D(x,y), \mathsf{ow}(xz) - \mathsf{ow}(z) = k \,\} \text{ and} \\ F_k &= \{\, \mathsf{nf}(z) \mid z \in D(x,y), \mathsf{ow}(zy) - \mathsf{ow}(z) = k \,\} \end{aligned}$$

are regular. Consequently, the language

$$\bigcup_{0 \leq k \leq |\pi(x)|} E_k \cap F_k$$

is regular. Since one can also show that $0 \leq \mathsf{ow}(xz) - \mathsf{ow}(z) \leq |\pi(x)|$ for $z \in D(x, y)$, this language equals the language of all words $\mathsf{nf}(z)$ with $z \in C(x, y)$. Hence $C(x, y)$ is the image wrt. the natural epimorphism η of a regular language and therefore rational.

8 Rational subsets

This section studies decision problems concerning rational subsets of $\mathcal{Q}$.

Let $w \in \Sigma^*$. Then, by Theorem 5.3, the number of left-divisors of $[w]$ in $\mathcal{Q}$ is at most $|w|^3$. This allows to define a DFA with $|w|^3$ many states that recognizes the language $[w] = \{\, u \in \Sigma^* \mid u \equiv w \,\}$. Even more, this DFA can be constructed in logarithmic space. This fact allows to reduce the problem below in logarithmic space to the intersection problem of NFAs. Hence we get the following result, where completeness follows since A^* embeds into $\mathcal{Q}$:

Theorem 8.1. *The following rational subset membership problem for $\mathcal{Q}$ is* NL*-complete:*

Input: A word $w \in \Sigma^*$ and an NFA $\mathcal{A}$ over Σ.
Question: Is there a word $v \in L(\mathcal{A})$ with $w \equiv v$?

We do not have a description of the submonoids of $\mathcal{Q}$, but we get the following embedding of the direct product of two free monoids.

Proposition 8.2. *Let $\mathcal{R} \subseteq \mathcal{Q}$ denote the submonoid generated by $\{[a], [ab], [\bar{b}], [\overline{a\bar{b}b}]\}$.*

(1) There exists an isomorphism α from $\{a, b\}^* \times \{c, d\}^*$ onto $\mathcal{R}$ with $\alpha((a, \varepsilon)) = [a]$, $\alpha((b, \varepsilon)) = [ab]$, $\alpha((\varepsilon, c)) = [\bar{b}]$, and $\alpha((\varepsilon, d)) = [\overline{a\bar{b}b}]$.
(2) If $\mathcal{S} \subseteq \mathcal{R}$ is recognizable in $\mathcal{R}$, then it is recognizable in $\mathcal{Q}$.

The proof makes heavy use of Theorem 4.1. This proposition implies in particular that rational transductions can be translated into rational subsets of $\mathcal{Q}$, resulting in the following undecidability results:

Theorem 8.3. *(1) The set of rational subsets of $\mathcal{Q}$ is not closed under intersection.*
(2) The emptiness of the intersection of two rational subsets of $\mathcal{Q}$ is undecidable.
(3) The universality of a rational subset of $\mathcal{Q}$ is undecidable.
Consequently, inclusion and equality of rational subsets are undecidable.
(4) The recognizability of a rational subset of $\mathcal{Q}$ is undecidable.

We sketch the proof of statement (3), the other claims are proved along similar lines: Let $S \subseteq \{a,b\}^* \times \{c,d\}^*$ be rational. Then $\alpha(S)$ is rational. Due to Prop. 8.2 (2), the set $\mathcal{R}$ is recognizable in $\mathcal{Q}$. Thus, $\mathcal{Q} \setminus \mathcal{R}$ is recognizable and therefore, since $\mathcal{Q}$ is finitely generated, rational. Consequently, $\alpha(S) \cup \mathcal{Q} \setminus \mathcal{R}$ is rational as well. This rational set equals $\mathcal{Q}$ if and only if $\alpha(S) = \mathcal{R}$, i.e., $S = \{a,b\}^* \times \{c,d\}^*$. But this latter question is undecidable by [1, Theorem 8.4(iv)].

9 Recognizable subsets

In this section, we aim to describe the recognizable subsets of $\mathcal{Q}$. Clearly, sets of the form $\pi^{-1}(L)$ or $\overline{\pi}^{-1}(L)$ for some regular $L \subseteq A^*$ as well as Boolean combinations thereof are recognizable. This does not suffice to produce all recognizable subsets: for instance, the singleton set $\{[\overline{a}a]\}$ is recognizable but any Boolean combination of inverse projections containing $[\overline{a}a]$ also includes $[a\overline{a}]$. However, we will see in the main result of this section, namely Theorem 9.4, that incorporating certain sets that can impose a simple restriction on relative positions of write and read symbols suffices to generate the recognizable sets as a Boolean algebra.

Recall that any $q \in \mathcal{Q}$ is completely determined by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$. Consequently, it would seem natural to incorporate sets which restrict the overlap width. Unfortunately, this does not work since the set of all $q \in \mathcal{Q}$ with $\mathsf{ow}(q) = k$ is not recognizable (for any $k \in \mathbb{N}$).

Nevertheless, the subsequent definition provides a slight variation of this idea which conduces to our purpose. To simplify notation, we say two elements $p, q \in \mathcal{Q}$ *have the same projections* and write $p \sim_\pi q$ if $\pi(p) = \pi(q)$ and $\overline{\pi}(p) = \overline{\pi}(q)$.

Definition 9.1. *For each $k \in \mathbb{N}$, the set $\Omega_k \subseteq \mathcal{Q}$ is given by*

$$\Omega_k = \{\, q \in \mathcal{Q} \mid \forall p \in \mathcal{Q}\colon p \sim_\pi q \;\&\; \mathsf{ow}(q) \le \mathsf{ow}(p) \le k \implies p = q \,\} \,.$$

Observe that $\mathcal{Q} = \Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \ldots$. Intuitively, for fixed projections $\pi(q)$ and $\overline{\pi}(q)$ the set Ω_k contains all q with $\mathsf{ow}(q) \ge k$ as well as the unique q with maximal $\mathsf{ow}(q) \le k$.

Example 9.2. (1) The queue action $q = [\overline{a}\overline{b}a\overline{a}ba]$ satisfies $\mathsf{ow}(q) = 1$ and hence $q \in \Omega_1$. The only $p \in \mathcal{Q}$ with $p \sim_\pi q$ and $\mathsf{ow}(p) \ge \mathsf{ow}(q)$ is $p = [a\overline{a}b\overline{b}a\overline{a}]$. Since $\mathsf{ow}(p) = 3$, this implies $q \in \Omega_2$ but $q \notin \Omega_3$.
(2) For every $k \ge 1$, we have $[(\overline{a}a)^k] \in \Omega_{k-1} \setminus \Omega_k$.
(3) All queue actions of the form $q = [u\overline{v}]$ with $u, v \in A^*$ satisfy $q \in \Omega_k$ for every $k \in \mathbb{N}$.

Remark 9.3. We know that $q \in \mathcal{Q}$ is uniquely described by $\pi(q)$, $\overline{\pi}(q)$, and $\mathsf{ow}(q)$. Somewhat surprisingly, we still have a unique description of q if we replace $\mathsf{ow}(q)$ by the maximal $k \in \mathbb{N}$ with $q \in \Omega_k$ or the fact that there is no such maximum.

The aforementioned main result characterizing the recognizable subsets of $\mathcal{Q}$ is the following.

Theorem 9.4. *For every subset $L \subseteq \mathcal{Q}$, the following are equivalent:*

(1) *L is recognizable,*
(2) *L is* wrw-recognizable*, i.e., the language $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$ is regular,*
(3) *$\eta^{-1}(L) \cap \overline{A}^*A^*\overline{A}^*$ is regular,*
(4) *L is* simple*, i.e., a Boolean combination of sets of the form $\pi^{-1}(R)$ or $\overline{\pi}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_k for $k \in \mathbb{N}$.*

The implication "(1)⇒(2)" is trivial. Regarding wrw-recognizability note that $L = \eta(\eta^{-1}(L) \cap A^*\overline{A}^*A^*)$ by Example 6.1, i.e., the language $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$ describes the set L completely. This is not the case if we replace $A^*\overline{A}^*A^*$ by $A^*\overline{A}^*$: The set $L = \{\, [\overline{a}^n a\overline{a}a^n] \mid n \geq 1 \,\}$ is not recognizable, since the set of its normal forms is not regular. However, $\eta^{-1}(L) \cap A^*\overline{A}^*$ is empty and hence regular.

Note that the implication"(4)⇒(1)" follows easily from the following result:

Proposition 9.5. *For each $k \in \mathbb{N}$, the set $\eta^{-1}(\Omega_k)$ is regular.*

The crucial point in its proof is the following characterization of the language $\eta(\Omega_k)$: $w \in \Sigma^*$ belongs $\eta^{-1}(\Omega_k)$ if and only if, for every $u \in A^{\leq k}$, one of the following holds:

1. u is no prefix of $\pi(w)$ or
2. u is no suffix of $\overline{\pi}(w)$ or
3. the i^{th} write symbol in w appears before the i^{th} of the last $|u|$ read symbols (for all $1 \leq i \leq |u|$).

As an illustration, $a\overline{a}b\overline{a}\overline{b}a\overline{a}a$ belongs to $\eta^{-1}(\Omega_3)$ and $a\overline{a}b\overline{a}\overline{b}\overline{a}aa$ does not. For every $u \in A^{\leq k}$ the language of words w satisfying one of the above three conditions is regular. Hence $\eta^{-1}(\Omega_k)$ is the intersection of finitely many regular languages and therefore regular.

The implication "(2)⇒(4)" of Theorem 9.4 is the following:

Proposition 9.6. *If $L \subseteq \mathcal{Q}$ is wrw-recognizable, then it is simple.*

Proof idea. Let k be the number of elements of the syntactic monoid of $\eta^{-1}(L) \cap A^*\overline{A}^*A^*$. Consider the following partition of L:

$$L = \left(L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k\right) \cup \left(L \cap \pi^{-1}\left(A^{\geq k}\right) \cap \Omega_k\right) \cup \bigcup_{0 \leq \ell < k} \left(L \cap \Omega_\ell \setminus \Omega_{\ell+1}\right).$$

One can show that all the parts are simple; we indicate how this is done for the first part, i.e., the set $L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k$:

Let $K = \eta^{-1}(L) \cap A^*\overline{A}^*A^*$ and $\phi\colon \Sigma^* \to M$ be a morphism recognizing K. We further consider the morphisms $\mu, \overline{\mu}\colon A^* \to M$ defined by $\mu(w) = \phi(w)$ and $\overline{\mu}(w) = \phi(\overline{w})$. We show the claim by establishing the equation

$$L \cap \pi^{-1}\left(A^{<k}\right) \cap \Omega_k = \bigcup_{\substack{u \in A^{<k}, m \in M \\ \mu(u)m \in \phi(K)}} \pi^{-1}(u) \cap \overline{\pi}^{-1}\left(\overline{\mu}^{-1}(m)\right) \cap \Omega_k \,.$$

Let X and Y denote the left and right hand side of this equation, respectively. Clearly, $X, Y \subseteq \pi^{-1}\left(A^{<k}\right) \cap \Omega_k$. Consider some $q \in \pi^{-1}\left(A^{<k}\right) \cap \Omega_k$. It suffices to show that $q \in X$ precisely if $q \in Y$.

To this end, let $u = \pi(q)$. Then $|u| < k$. Using $u \in \Omega_k$, one can show that there is $p \in \mathcal{Q}$ such that $q = [u]\,p$. Clearly, $\pi(p) = \varepsilon$, i.e., $p = [\overline{y}]$ for some $y \in A^*$. Notice that $q = [u\overline{y}]$. Altogether,

$$\begin{aligned} q \in X \quad &\iff \quad q = [u\overline{y}] \in L \\ &\iff \quad \phi(u\overline{y}) = \mu\,(u)\;\overline{\mu}\,(\overline{\pi}(q)) \in \phi(K) \quad \iff \quad q \in Y \,. \end{aligned}$$

The simplicity of the other sets is shown using similar arguments.

Since the implication "(1)⇒(2)" in Theorem 9.4 is trivial, we have the equivalence of (1), (2), and (4). Claim (3) can be added using duality arguments.

10 Thurston-automaticity

Many groups of interest in combinatorial group theory turned out to be Thurston-automatic [4]. The more general concept of a Thurston-automatic semigroup was introduced in [3]. In this chapter, we prove that the monoid of queue-actions $\mathcal{Q}$ does not fall into this class.

Let M be a monoid, Γ an alphabet, $\theta\colon \Gamma^+ \to M$ a semigroup morphism, and $L \subseteq \Gamma^+$. The triple (Γ, θ, L) is an *automatic structure* for the monoid M if θ maps L bijectively onto M, if the language L is regular and if the relations

$$L_a = \left\{ \, (u, v) \in L^2 \;\middle|\; \theta(ua) = \theta(v) \, \right\} \subseteq L^2$$

are synchronously rational (i.e., accepted by a synchronous transducer, cf. [1, 8]) for all $a \in \Gamma$.[3] A monoid is *Thurston-automatic* if it has some automatic structure.

Theorem 10.1. *The monoid of queue actions $\mathcal{Q}$ is not Thurston-automatic.*

Proof idea. Suppose $\mathcal{Q}$ is Thurston-automatic. By [7], there exists an automatic structure $(\Sigma \cup \{\iota\}, \theta, L)$ for $\mathcal{Q}$ with $\theta(a) = [a]$, $\theta(\overline{a}) = [\overline{a}]$ for $a \in A$, and $\theta(\iota) = [\varepsilon]$. For $m, n \in \mathbb{N}$, let $u_{m,n} \in L$ be the unique word with $\theta(u_{m,n}) = [\overline{a}^m a^n]$. By Theorem 5.3, there are precisely $\min(m, n) + 1$ many $q \in Q$ with $[\overline{a}^m a^n \overline{b}] = q[\overline{b}]$. It follows that this is the number of words $w \in L$ with $u_{m,n}\overline{b} \equiv w\overline{b}$. Since the set of pairs $(u_{m,n}, w)$ satisfying this equation (with $m, n \in \mathbb{N}$ and $w \in (\Sigma \cup \{\iota\})^*$) is synchronously rational

[3] This is not the original definition from [3], but it is equivalent by [3, Prop. 5.4].

[3], one can construct a nondeterministic finite automaton $\mathcal{A}$ with $\min(m, n) + 1$ many runs on any word of the form $\overline{a}^m a^n$. This then leads to a contradiction. $\square$

Recently, the notion of an automatic group has been extended to that of Cayley graph automatic groups [11]. This notion can easily be extended to monoids. It is not clear whether the monoid $\mathcal{Q}$ is Cayley graph automatic. A way to disprove this would be to show that the elementary theory of its Cayley graph is undecidable.

Note that $\mathcal{Q}$ is not automatic in the sense of Khoussainov and Nerode [12]: This is due to the fact that $\eta(A^*)$ is isomorphic to A^* and an element of $\mathcal{Q}$ is in $\eta(A^*)$ if and only if it cannot be written as $r\overline{a}s$ for $r, s \in \mathcal{Q}$ and $a \in A$. Hence, using the $\overline{a}$ for $a \in A$ as parameters, A^* is interpretable in first order logic in $\mathcal{Q}$. Therefore, since A^* is not automatic in this sense [2], neither is $\mathcal{Q}$ [12].

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