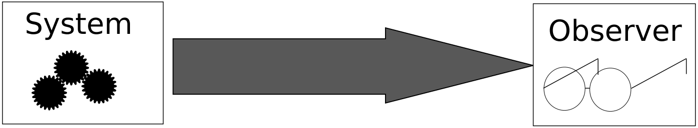
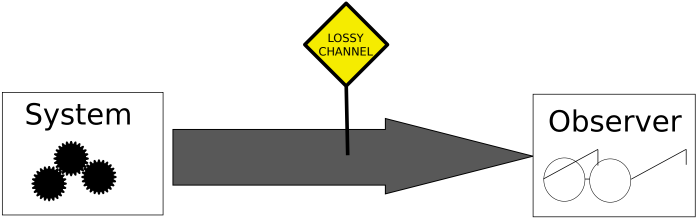
Downward Closures of Indexed Languages

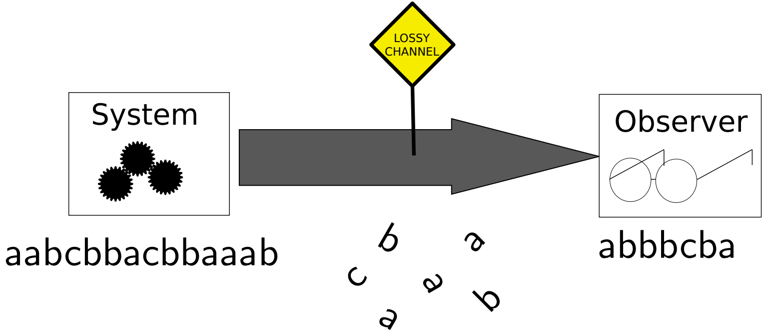
Georg Zetzsche

Technische Universität Kaiserslautern

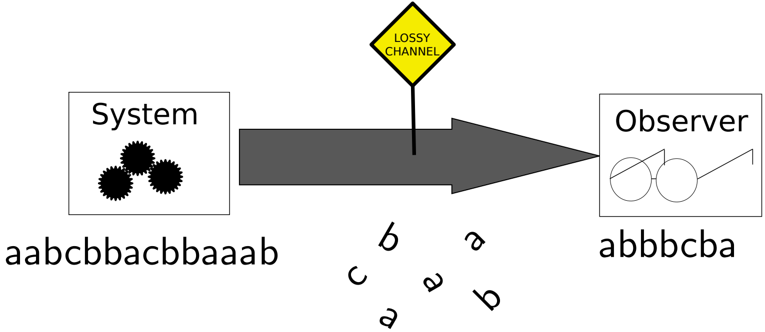
HOPA 2015


System
Observer


System
LOSSY
CHANNEL
Observer



$\underline{a}ab\underline{c}bb\underline{a}cb\underline{b}a\underline{aab}$



aabcbbacbbaaab

Downward Closures

- $u \preccurlyeq v$: u is a subsequence of v
- $L\!\downarrow = \{u \in X^* \mid \exists v \in L \colon u \preccurlyeq v\}$
- Observer sees precisely $L\!\downarrow$

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Problem

- Finite automaton for $L\downarrow$ exists for every L.
- How can we compute it?

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Downward closures are computable for stacked counter automata.

- Weak form of stack nesting
- Adding Counters

Negative results

Theorem (Gruber, Holzer, Kutrib 2009)

Downward closures are not computable when infinity or emptiness are undecidable.

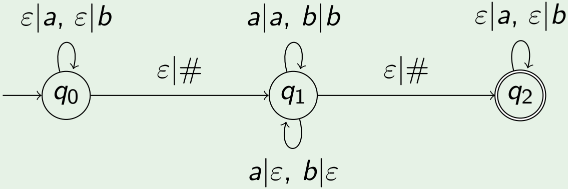
Theorem (Mayr 2003)

The reachability set of lossy channel systems is not computable.

Theorem (Z. 2015)

Downward closures are computable for indexed languages, i.e. for second-order pushdown automata.

Example (Transducer)



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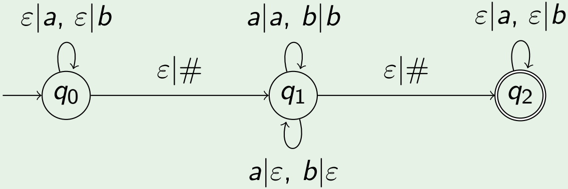

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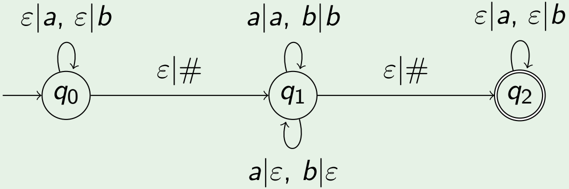

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $T \subseteq X^* \times Y^*$ and language $L \subseteq Y^*$, let

$$TL = \{y \in X^* \mid \exists x \in L : (x, y) \in T\}$$

Fact (Aho 1968)

For every indexed language L and rational transduction T, the language TL is indexed as well.

Theorem (Z. 2015)

Let $\mathcal{C}$ be a language class that is closed under rational transductions. Then downward closures are computable for $\mathcal{C}$ if and only if the following problem is decidable:

Given	*A language $L \subseteq a_1^* \cdots a_n^*$ in $\mathcal{C}$*
Question	*Does $L{\downarrow}$ equal $a_1^* \cdots a_n^*$?*

Theorem (Jullien 1969, Abdulla et. al. 2004)

Every language $L\downarrow$ can be written as a finite union of sets of the form

$$Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^*,$$

where $x_1,\ldots,x_n$ are letters and $Y_0,\ldots,Y_n$ are alphabets.

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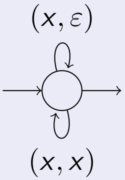
$L\!\downarrow$ is in $\mathcal{C}$:



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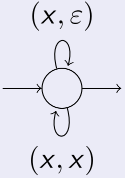

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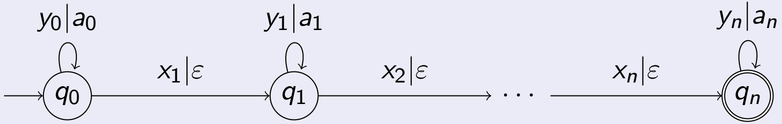

y_i: word containing each letter of Y_i once.

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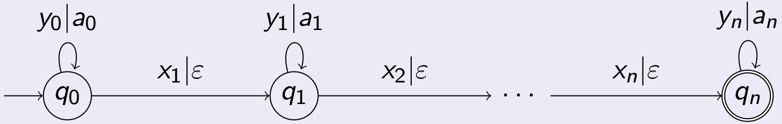

y_i: word containing each letter of Y_i once. Then:

$$T(L{\downarrow}){\downarrow} = a_0^*\cdots a_n^* \quad \text{iff} \quad Y_0^*\{x_1,\varepsilon\}Y_1^*\cdots\{x_n,\varepsilon\}Y_n^* \subseteq L{\downarrow}$$

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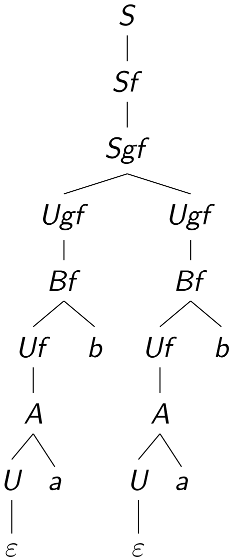

Application to Indexed Languages

Given: indexed grammar G with $L = L(G) \subseteq a_1^* \cdots a_n^*$, wlog $L = L\downarrow$.

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Then, $L(G)\!\downarrow = a_1^* \cdots a_n^*$ iff $L(G_D)\!\downarrow = a_1^* \cdots a_n^*$ for some D.

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Idea

Instead of unfolding a_i-subtree with root Au, $u \in I^*$, apply transducer to u

Preserving $L(G)\downarrow = a_1^* \cdots a_n^*$

For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G\colon NI^* \to \mathbb{N}\{\infty\}$ be

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- Only one nonterminal occurrence for transducer

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For transduction $T \subseteq NI^* \times a_i^*$, let $f_T, f_G \colon NI^* \to \mathbb{N}\{\infty\}$ be

$$f_T(Au) = \sup\{|v| \mid (u,v) \in T\}$$
$$f_G(Au) = \sup\{|v| \mid v \in a_i^*,\ Au \Rightarrow_G^* v\}$$

Proposition

For each indexed grammar G, one can construct a rational transduction T with $f_T \approx f_G$.

$f \approx g$: f is unnounded on the same subsets as g

Step 2: Apply transducer

- Instead of unfolding a_i-subtrees, $a_i \notin D$, apply transducer to index word.
- Only one nonterminal occurrence for transducer

$\Rightarrow$ Bound on nonterminal occurrences, "breadth-bounded"

Remaining problem

- Given: Breadth-bounded indexed grammar G, $L(G) \subseteq a_1^* \cdots a_n^*$
- Does $L(G){\downarrow} = a_1^* \cdots a_n^*$?

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Breadth-bounded indexed grammars have effectively semilinear Parikh images.

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Step 3:

Proposition

Breadth-bounded indexed grammars have effectively semilinear Parikh images.

Then, it is clearly decidable whether $L(G){\downarrow} = a_1^* \cdots a_n^*$.

Thank you for your attention!