Valence automata as a generalization of automata with storage

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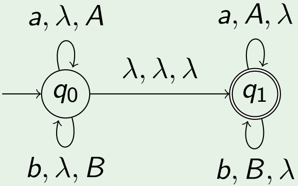
D-CON 2013

Example (Pushdown automaton)



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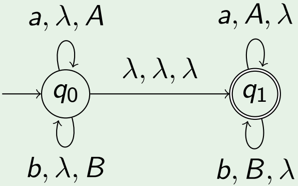

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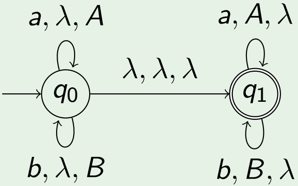

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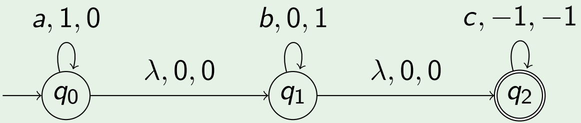

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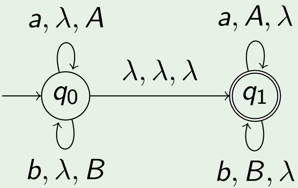

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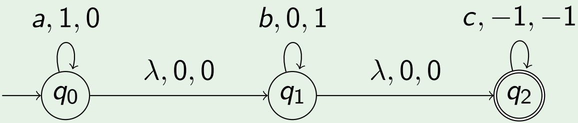
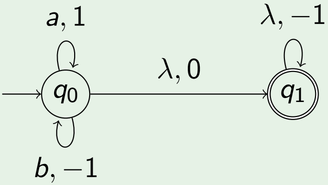


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



$$L = \{w \in \{a,b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for any prefix } p \text{ of } w\}$$

Automata models that extend finite automata by some storage mechanism:

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- Blind counter automata
- Partially blind counter automata
- Turing machines

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Each storage mechanism consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \to S$

Model	States	Operations
Pushdown automata	$S = \Gamma^*$	$\mathrm{push}_a : w \mapsto wa,\ a \in \Gamma$ $\mathrm{pop}_a : wa \mapsto w,\ a \in \Gamma$

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Observation

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \mathrm{id}$$

Definition

A *monoid* is

- a set M together with
- an associative binary operation $\cdot : M \times M \to M$ and
- a neutral element $1 \in M$ ($a1 = 1a = a$ for any $a \in M$).

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Storage mechanisms as monoids

- Let S be a set of states and $\alpha_1, \ldots, \alpha_n : S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid M.
- The identity map is the neutral element of M.
- M is a decription of the storage mechanism.

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Common generalization: Valence Automata

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  - q_0 is the initial state,
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  - $m_1 \cdots m_n = 1$.

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$\mathrm{detVA}(M)$ languages accepted by deterministic valence automata over M.

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Theorem

The following statements are equivalent:

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2. $\mathsf{detVA}(M) = \mathsf{VA}(M)$.
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(1) $\Leftrightarrow$ (3) has been shown independently by Elaine Render in 2010.

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Lemma (Dichotomy)

For each monoid M, exactly one of the following holds:

1. $\mathsf{R}(M) = \mathsf{L}(M) = \mathsf{J}(M)$ *is a finite group.*
2. *There are infinite subsets $S \subseteq \mathsf{R}(M)$, $S' \subseteq \mathsf{L}(M)$ such that*
   - $\overline{\mathsf{R}}(s) \cap \overline{\mathsf{R}}(t) = \varnothing$ *for any $s, t \in S$ and*
   - $\overline{\mathsf{L}}(s) \cap \overline{\mathsf{L}}(t) = \varnothing$ *for any $s, t \in S'$.*

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Important problem

- When can λ-transitions be eliminated?
- Without λ-transitions, decide membership using exponential number of queries to the word problem.
- Elimination can be regarded as a precomputation.

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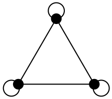
Intuition

To each graph Γ, we associate a monoid $\mathbb{M}\Gamma$:

- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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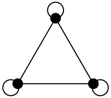

$\mathbb{Z}^3$

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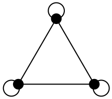

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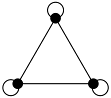

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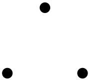

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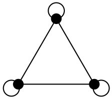

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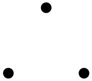

$\mathbb{B} * \mathbb{B} * \mathbb{B}$

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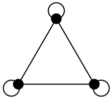

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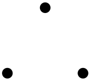

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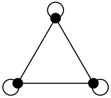

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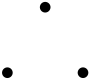

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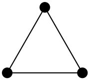

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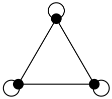

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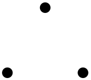

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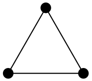

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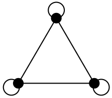

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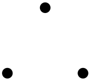

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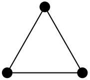

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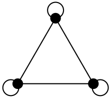

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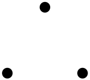

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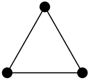

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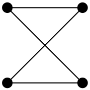

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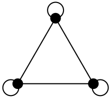

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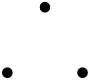

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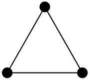

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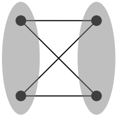

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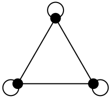

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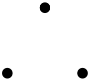

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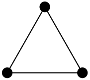

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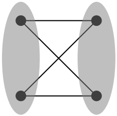

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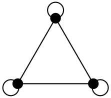
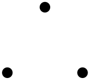
Examples



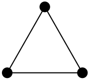
$\mathbb{Z}^3$

Blind multicounter



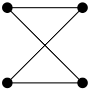
$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown



$\mathbb{B}^3$

Partially blind multicounter



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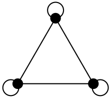
Infinite tape (TM)

Examples



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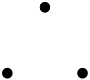
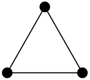


$\mathbb{B} * \mathbb{B} * \mathbb{B}$

Pushdown



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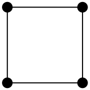

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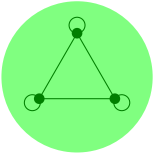

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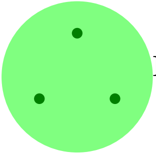

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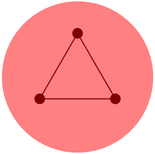
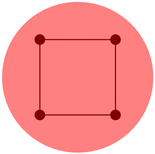
Pushdown



$\mathbb{B}^3$

Partially blind multicounter



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Infinite tape (TM)

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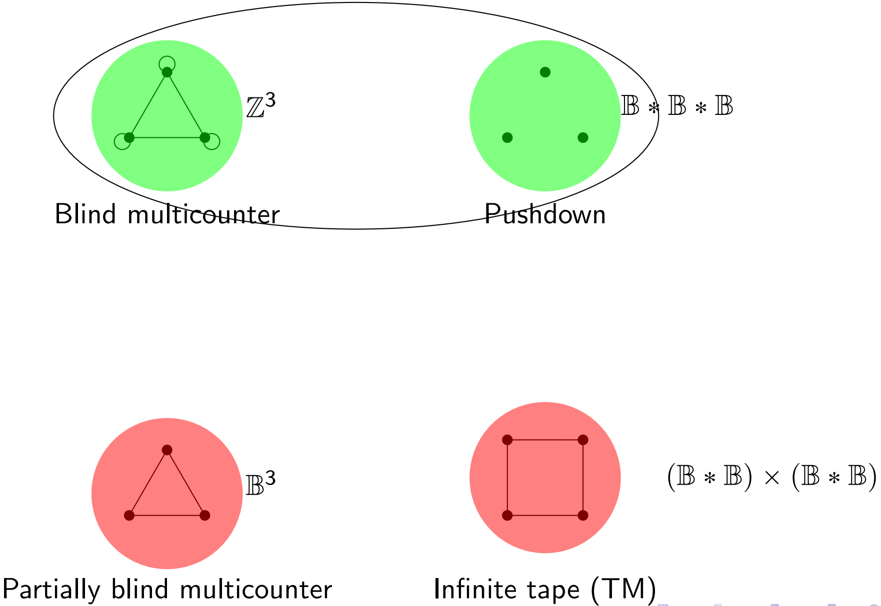

$\mathbb{Z}^3$
$\mathbb{B} * \mathbb{B} * \mathbb{B}$
Blind multicounter
Pushdown
$\mathbb{B}^3$
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Partially blind multicounter
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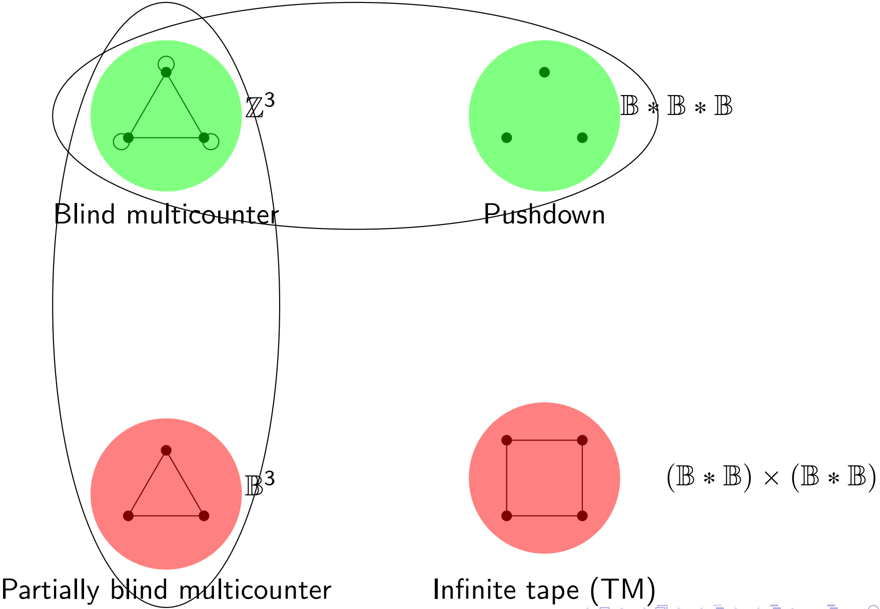
Examples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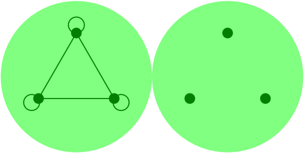

Theorem

Let Γ be a graph such that

- *between any two looped vertices, there is an edge and*
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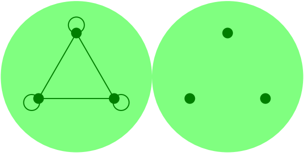

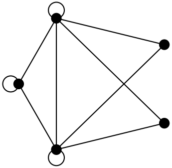

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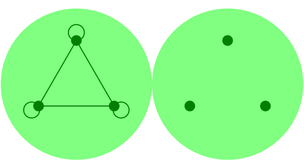

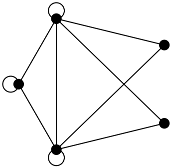

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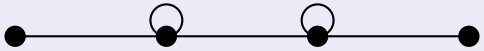
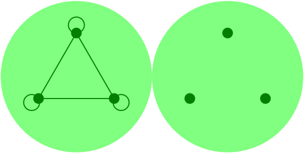
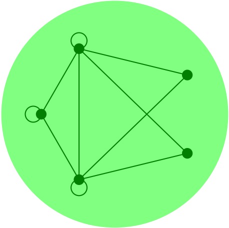


as an induced subgraph.





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as an induced subgraph.

By reduction to an undecidable problem from group theory, we obtain:

Lemma

Let Γ *be a graph with*

as an induced subgraph. Then $\mathsf{VA}(\mathbb{M}\Gamma)$ *contains an undecidable language. Hence,* $\mathsf{VA}^{+}(\mathbb{M}\Gamma) \subsetneq \mathsf{VA}(\mathbb{M}\Gamma)$.

Definition

Let $\mathcal{C}$ be the smallest class of monoids such that

- $1 \in \mathcal{C}$
- if $M \in \mathcal{C}$, then $M \times \mathbb{Z} \in \mathcal{C}$
- if $M \in \mathcal{C}$, then $M * \mathbb{B} \in \mathcal{C}$

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Lemma

Let Γ be a graph such that

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-  *does not appear as an induced subgraph*

Then, $\mathbb{M}\Gamma \in \mathcal{C}$.

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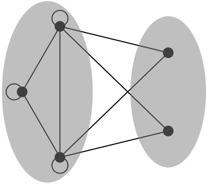
Interpretation of $\mathcal{C}$

$\mathcal{C}$ corresponds to the class of storage mechanisms obtained by

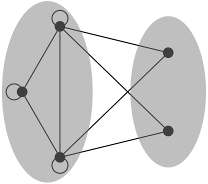
- adding a blind counter ($M \times \mathbb{Z}$) and
- building stacks ($M * \mathbb{B}$).



Proof.

- Let $\Gamma = (V, E)$, $V = L \cup U$, looped and unlooped vertices.
- For each $x \in L$, let $\nu(x) = N(x) \cap U$

$\square$



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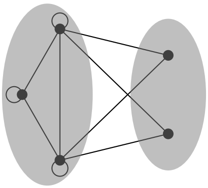

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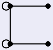
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.

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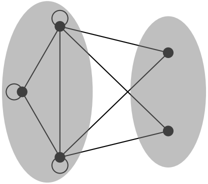

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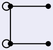
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.
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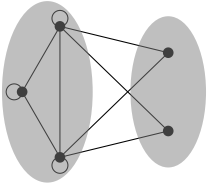

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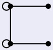
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.
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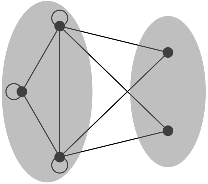
□



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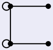
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.
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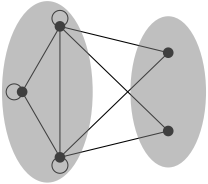

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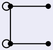
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.
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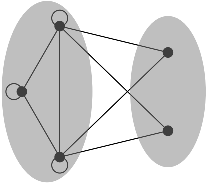

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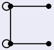
- Since  is not an induced subgraph, $\leqslant$ is a total preorder.
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□

Elimination of λ-transitions

Definition

A subset $S \subseteq M$ is called *rational* if it is the homomorphic image of a regular language.

Elimination of λ-transitions

Approach:

- Between a given pair of non-λ-transitions, the set of $x \in M$ applied in between is a rational set.
- Transform the automaton so as to simulate the application of a rational set in one step.

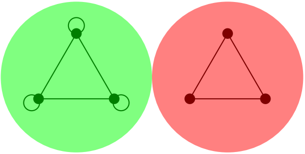
Proof ingredients

Ingredients

- Semilinearity of languages in $\mathsf{VA}(\mathbb{M}\Gamma)$
- Normal form for rational subsets of $\mathbb{M}\Gamma$

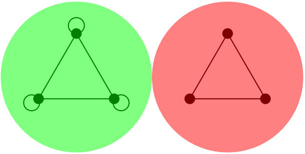
Construction for $\mathsf{VA}^+(\mathbb{M}\Gamma) = \mathsf{VA}(\mathbb{M}\Gamma)$

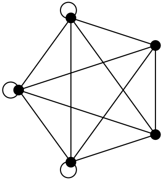
- Actual construction quite involved.
- Stronger claim to make induction work.
- Separate constructions for $\mathbb{B}$, $M \times \mathbb{Z}$, and $M * \mathbb{B}$.
- Representations of rational sets are encoded into the state or the monoid elements.



Theorem

Let Γ be a graph such that between any two distinct vertices, there is an edge.





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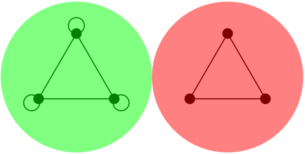

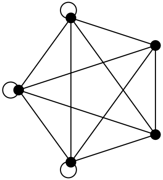

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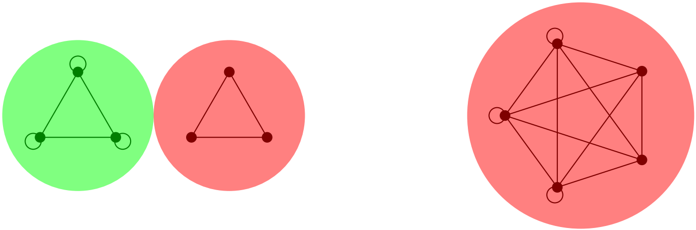

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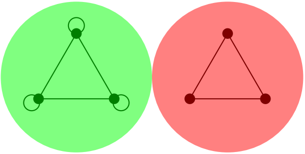

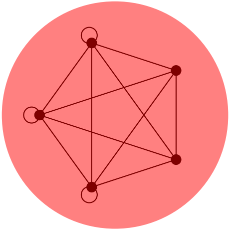

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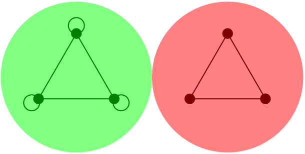
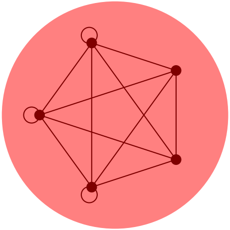
$$\mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \quad \textit{iff} \quad r \leqslant 1.$$





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$$\mathsf{VA}(\mathbb{B}^r \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B}^r \times \mathbb{Z}^s) \quad \textit{iff} \quad r \leqslant 1.$$

Observation

$\mathsf{VA}(\mathbb{B} \times \mathbb{Z}^s) = \mathsf{VA}^+(\mathbb{B} \times \mathbb{Z}^s)$ already follows from the first theorem.

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Work in Progress

- For which storage mechanisms do we have a Parikh theorem?

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