SEMILINEARITY AND CONTEXT-FREENESS OF LANGUAGES ACCEPTED BY VALENCE AUTOMATA

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Abstract. Valence automata are a generalization of various models of automata with storage. Here, each edge carries, in addition to an input word, an element of a monoid. A computation is considered valid if multiplying the monoid elements on the visited edges yields the identity element. By choosing suitable monoids, a variety of automata models can be obtained as special valence automata.

This work is concerned with the accepting power of valence automata. Specifically, we ask for which monoids valence automata can accept only context-free languages or only languages with semilinear Parikh image, respectively.

First, we present a characterization of those graph products (of monoids) for which valence automata accept only context-free languages. Second, we provide a necessary and sufficient condition for a graph product of copies of the bicyclic monoid and the integers to yield only languages with semilinear Parikh image when used as a storage mechanism in valence automata. Third, we show that all languages accepted by valence automata over torsion groups have a semilinear Parikh image.

1. Introduction

A valence automaton is a finite automaton in which each edge carries, in addition to an input word, an element of a monoid. A computation is considered valid if multiplying the monoid elements on the visited edges yields the identity element. By choosing suitable monoids, one can obtain a wide range of automata with storage mechanisms as special valence automata. Thus, they offer a framework for generalizing insights about automata with storage. For examples of automata as valence automata, see [5, 22].

In this work, we are concerned with the accepting power of valence automata. That is, we are interested in relationships between the structure of the monoid representing the storage mechanism and the class of languages accepted by the corresponding valence automata. On the one hand, we address the question for which monoids valence automata accept only *context-free* languages. Since the context-free languages constitute a very well-understood class, insights in this direction promise to shed light on the acceptability of languages by transferring results about context-free languages.

A very well-known result on context-free languages is Parikh's Theorem [17], which states that the Parikh image (that is, the image under the canonical morphism onto the free commutative monoid) of each context-free language is semilinear (in this case, the language itself is also called semilinear). It has various applications in proving that certain languages are not context-free and its effective nature (one can actually compute the semilinear representation) allows it to be used in decision procedures for numerous problems (see [15] for an example from group theory and [11] for others). It is therefore our second goal to gain understanding about which monoids cause the corresponding valence automata to accept only languages with a *semilinear Parikh image*.

Our contribution is threefold. First, we obtain a characterization of those graph products (of monoids) whose corresponding valence automata accept only context-free languages. Graph products are a generalization of the free and the direct product in the sense that for each pair of participating factors, it can be specified whether they should commute in the product. Since valence automata over a group accept only context-free languages if and only if the group's word problem (and hence the group itself) can be described by a context-free grammar, such a characterization had already been available for groups in a result by Lohrey and Sénizergues [13]. Therefore, our characterization is in some sense an extension of Lohrey and Sénizergues' to monoids.

Second, we present a necessary and sufficient condition for a graph product of copies of the bicyclic monoid and the integers to yield, when used in valence automata, only languages with semilinear Parikh image. Although this is a smaller class of monoids than arbitrary graph products, it still covers a number of storage mechanisms found in the literature, such as pushdown automata, blind multicounter automata, and partially blind multicounter automata (see [22] for more information). Hence, our result is a generalization of various semilinearity results about these types of automata.

Third, we show that every language accepted by a valence automaton over a torsion group has a semilinear Parikh image. On the one hand, this is particularly interesting because of a result by Render [18], which states that for every monoid M, the languages accepted by valence automata over M either (1) coincide with the regular languages, (2) contain the blind one-counter languages, (3) contain the partially blind one-counter languages, or (4) are those accepted by valence automata over an infinite torsion group (which is not locally finite). Hence, our result establishes a strong language theoretic property in the fourth case and thus contributes to completing the picture of language classes that can arise from valence automata.

On the other hand, Lohrey and Steinberg [15] have used the fact that for certain groups, valence automata accept only semilinear languages (in different terms, however) to obtain decidability of the rational subset membership problem. However, their procedures require that the semilinear representation can be obtained effectively. Since there are torsion groups where even the word problem is undecidable [1], our result yields examples of groups that have the semilinearity property but which do not permit the computation of a corresponding representation. Our proof is based on well-quasi-orderings (see, e.g., [12]).

2. Basic notions

We assume that the reader has some basic knowledge on formal languages and monoids. In this section, we will fix some notation and introduce basic concepts.

A *monoid* is a set M together with an associative operation and a neutral element. Unless defined otherwise, we will denote the neutral element of a monoid by 1 and its operation by juxtaposition. That is, for a monoid M and elements

$a, b \in M$, $ab \in M$ is their product. In each monoid M, we have the submonoids

$$\begin{aligned} \mathsf{R}(M) &= \{a \in M \mid \exists b \in M : ab = 1\}, \\ \mathsf{L}(M) &= \{a \in M \mid \exists b \in M : ba = 1\}. \end{aligned}$$

When using a monoid M as part of a control mechanism, the subset

$$\mathsf{J}(M) = \{a \in M \mid \exists b, c \in M : bac = 1\}$$

plays an important role[1] A *subgroup* of a monoid is a subset that is closed under the operation and is a group.

Let Σ be a fixed countable set of abstract symbols, the finite subsets of which are called *alphabets*. For a set of symbols $X \subseteq \Sigma$, we will write X^* for the set of words over X. The empty word is denoted by $\lambda \in X^*$. Together with concatenation as its operation, X^* is a monoid. Given an alphabet X and a monoid M, subsets of X^* and $X^* \times M$ are called *languages* and *transductions*, respectively. A *family* is a set of languages that is closed under isomorphism and contains at least one non-trivial member. For a transduction $T \subseteq X^* \times Y^*$ and a language $L \subseteq X^*$, we write $T(L) = \{v \in Y^* \mid \exists u \in L : (u, v) \in T\}$. For any finite subset $S \subseteq M$ of a monoid, let X_S be an alphabet in bijection with S. Let $\varphi_S : X_S^* \to M$ be the morphism extending this bijection. Then the set $\{w \in X_S^* \mid \varphi_S(w) = 1\}$ is called the *identity language of M with respect to S*.

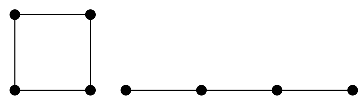
Let $\mathcal{F}$ be a family of languages. An *$\mathcal{F}$-grammar* is a quadruple $G = (N, T, P, S)$ where N and T are disjoint alphabets and $S \in N$. P is a finite set of pairs (A, M) with $A \in N$ and $M \subseteq (N \cup T)^*$, $M \in \mathcal{F}$. In this context, a pair $(A, M) \in P$ will also be denoted by $A \to M$. We write $x \Rightarrow_G y$ if $x = uAv$ and $y = uwv$ for some $u, v, w \in (N \cup T)^*$ and $(A, M) \in P$ with $w \in M$. The *language generated by G* is $L(G) = \{w \in T^* \mid S \Rightarrow_G^* w\}$. A language L is called *algebraic over $\mathcal{F}$* if there is an $\mathcal{F}$-grammar G such that $L = L(G)$. The family of all languages that are algebraic over $\mathcal{F}$ is called the *algebraic extension* of $\mathcal{F}$. The algebraic extension of the family of finite languages is denoted CF, its members are called *context-free*.

Given an alphabet X, we write $X^\oplus$ for the set of maps $\alpha : X \to \mathbb{N}$. Elements of $X^\oplus$ are called *multisets*. By way of pointwise addition, written $\alpha + \beta$, $X^\oplus$ is a monoid. The *Parikh mapping* is the mapping $\Psi : X^* \to X^\oplus$ such that $\Psi(w)(x)$ is the number of occurrences of x in w for every $w \in X^*$ and $x \in X$.

Let A be a (not necessarily finite) set of symbols and $R \subseteq A^* \times A^*$. The pair (A, R) is called a *(monoid) presentation*. The smallest congruence of A^* containing R is denoted by $\equiv_R$ and we will write $[w]_R$ for the congruence class of $w \in A^*$. The *monoid presented by* (A, R) is defined as $A^*/{\equiv_R}$. Note that since we did not impose a finiteness restriction on A, every monoid has a presentation.

Let M be a monoid. An *automaton over* M is a tuple $A = (Q, M, E, q_0, F)$, in which Q is a finite set of *states*, E is a finite subset of $Q \times M \times Q$ called the set of *edges*, $q_0 \in Q$ is the *initial state*, and $F \subseteq Q$ is the set of *final states*. The *step relation* $\Rightarrow_A$ of A is a binary relation on $Q \times M$, for which $(p, a) \Rightarrow_A (q, b)$ if and only if there is an edge (p, c, q) such that $b = ac$. The set *generated by* A is then $S(A) = \{a \in M \mid \exists q \in F : (q_0, 1) \Rightarrow_A^* (q, a)\}$. A set $R \subseteq M$ is called *rational* if it can be written as $R = S(A)$ for some automaton A over M. Rational languages

[1]It should be noted that $\mathsf{R}(M)$, $\mathsf{L}(M)$, and $\mathsf{J}(M)$ are the $\mathcal{R}$-, $\mathcal{L}$-, and $\mathcal{J}$-class, respectively, of the identity and hence are important concepts in the theory of semigroups [8].



(A) (B)

FIGURE 1. Graphs C_4 and P_4.

are also called *regular*, the corresponding class is denoted REG. A class $\mathcal{C}$ for which $L \in \mathcal{C}$ implies $T(L) \in \mathcal{C}$ for every rational transduction T is called a *full trio*.

For $n \in \mathbb{N}$ and $\alpha \in X^{\oplus}$, we use $n\alpha$ to denote $\alpha + \cdots + \alpha$ (n summands). A subset $S \subseteq X^{\oplus}$ is *linear* if there are elements $\alpha_0, \ldots, \alpha_n$ such that $S = \{\alpha_0 + \sum_{i=1}^{n} m_i \alpha_i \mid m_i \in \mathbb{N},\ 1 \leq i \leq n\}$. A set $S \subseteq C$ is called *semilinear* if it is a finite union of linear sets. In slight abuse of terminology, we will sometimes call a language L semilinear if the set $\Psi(L)$ is semilinear.

A *valence automaton over* M is an automaton A over $X^* \times M$, where X is an alphabet. Instead of $A = (Q, X^* \times M, E, q_0, F)$, we then also write $A = (Q, X, M, E, q_0, F)$ and for an edge $(p, (w, m), q) \in E$, we also write (p, w, m, q). The *language accepted by* A is defined as $L(A) = \{w \in X^* \mid (w, 1) \in S(A)\}$. The class of languages accepted by valence automata over M is denoted by $\mathsf{VA}(M)$. It is well-known that $\mathsf{VA}(M)$ is the smallest full trio containing every identity language of M (see, for example, [10]).

A *graph* is a pair $\Gamma = (V, E)$ where V is a finite set and $E \subseteq \{S \subseteq V \mid 1 \leq |S| \leq 2\}$. The elements of V are called *vertices* and those of E are called *edges*. If $\{v\} \in E$ for some $v \in V$, then v is called a *looped* vertex, otherwise it is *unlooped*. A *subgraph* of Γ is a graph (V', E') with $V' \subseteq V$ and $E' \subseteq E$. Such a subgraph is called *induced (by V')* if $E' = \{S \in E \mid S \subseteq V'\}$, i.e. E' contains all edges from E incident to vertices in V'. By $\Gamma \setminus \{v\}$, for $v \in V$, we denote the subgraph of Γ induced by $V \setminus \{v\}$. Given a graph $\Gamma = (V, E)$, its *underlying loop-free graph* is $\Gamma' = (V, E')$ with $E' = E \cap \{S \subseteq V \mid |S| = 2\}$. For a vertex $v \in V$, the elements of $N(v) = \{w \in V \mid \{v, w\} \in E\}$ are called *neighbors* of v. Moreover, a *clique* is a graph in which any two distinct vertices are adjacent. A *simple path of length n* is a sequence $x_1, \ldots, x_n$ of pairwise distinct vertices such that $\{x_i, x_{i+1}\} \in E$ for $1 \leq i < n$. If, in addition, we have $\{x_n, x_1\} \in E$, it is called a *cycle*. Such a cycle is called *induced* if $\{x_i, x_j\} \in E$ implies $|i - j| = 1$ or $\{i, j\} = \{1, n\}$. A loop-free graph $\Gamma = (V, E)$ is *chordal* if it does not contain an induced cycle of length ≥ 4. It is well-known that every chordal graph contains a vertex whose neighborhood is a clique [4]. By C_4 and P_4, we denote the cycle of length 4 and the simple path of length 4, respectively (see figures 1a and 1b). A loop-free graph is called a *transitive forest* if it is the disjoint union of comparability graphs of rooted trees. A result by Wolk [20] states that a graph is a transitive forest if and only if it contains neither C_4 nor P_4 as an induced subgraph.

Let $\Gamma = (V, E)$ be a loop-free graph and M_v a monoid for each $v \in V$ with a presentation (A_v, R_v) such that the A_v are pairwise disjoint. Then the *graph product* $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ is the monoid given by the presentation (A, R), where $A = \bigcup_{v \in V} A_v$ and

$$R = \{(ab, ba) \mid a \in A_v, b \in A_w, \{v, w\} \in E\} \cup \bigcup_{v \in V} R_v.$$

Note that for each $v \in V$, there is a map $\varphi_v : M \to M_v$ such that φ_v is the identity map on M_v. When $V = \{0,1\}$ and $E = \emptyset$, we also write $M_0 * M_1$ for M and call this the *free product* of M_0 and M_1. Given a subset $U \subseteq V$, we write $M{\restriction_U}$ for the product $\mathbb{M}(\Gamma', (M_v)_{v\in U})$, where Γ' is the subgraph induced by U.

By $\mathbb{B}$, we denote the monoid presented by (A, R) with $A = \{x, \bar{x}\}$ and $R = (x\bar{x}, \lambda)$. The elements $[x]_R$ and $[\bar{x}]_R$ are called its *positive* and *negative generator*, respectively. The set D_1 of all $w \in \{x, \bar{x}\}^*$ with $[w]_R = [\lambda]_R$ is called the *Dyck language*. The group of integers is denoted $\mathbb{Z}$. Here, we call $1 \in \mathbb{Z}$ its *positive* and $-1 \in \mathbb{Z}$ its *negative generator*.

Let $\Gamma = (V, E)$ be a (not necessarily loop-free) graph. Furthermore, for each $v \in V$, let M_v be a copy of $\mathbb{B}$ if v is an unlooped vertex and a copy of $\mathbb{Z}$ if v is looped. If Γ^- is obtained from Γ by removing all loops, we write $\mathbb{M}\Gamma$ for the graph product $\mathbb{M}(\Gamma^-, (M_v)_{v\in V})$. For information on valence automata over monoids $\mathbb{M}\Gamma$, see [22].

For $i \in \{0,1\}$, let M_i be a monoid and let $\varphi_i : N \to M_i$ be an injective morphism. Let $\equiv$ be the smallest congruence in $M_0 * M_1$ such that $\varphi_0(a) \equiv \varphi_1(a)$ for every $a \in N$. Then the monoid $(M_0 * M_1)/\equiv$ is denoted by $M_0 *_N M_1$ and called a *free product with amalgamation*.

3. Auxiliary Results

In this section, we present auxiliary results that are used in later sections. In the following, we will call a monoid M an *FRI-monoid* (or say that M has the FRI-property) if for every finitely generated submonoid N of M, the set $\mathsf{R}(N)$ is finite. In [18] and independently in [21], the following was shown.

Theorem 1. *For each monoid M, the following are equivalent:*

(1) M is an FRI-monoid.
(2) $\mathsf{VA}(M) = \mathsf{REG}$.

The first two lemmas state well-known facts from semigroup theory for which we provide short proofs for the sake of accessibility.

Lemma 2. *For each monoid M, exactly one of the following holds:*

(1) $\mathsf{J}(M)$ is a group,
(2) M contains a copy of $\mathbb{B}$ as a submonoid.

Proof. If $\mathsf{R}(M) = \mathsf{L}(M)$, then $\mathsf{J}(M) = \mathsf{R}(M) = \mathsf{L}(M)$ and hence $\mathsf{J}(M)$ is a group. Otherwise, if $x \in \mathsf{R}(M)\setminus\mathsf{L}(M)$ with $xy = 1$, it can be verified straightforwardly that the submonoid generated by x and y is isomorphic to $\mathbb{B}$. If $\mathsf{L}(M) \setminus \mathsf{R}(M) \neq \emptyset$, we can proceed analogously. The two cases are mutually exclusive, since in the second case, we have $xy = 1$ and $yx \neq 1$, where x and y are the positive and negative generator of $\mathbb{B}$, respectively. This, however, cannot happen in a group. $\square$

Lemma 3. *For each monoid M, exactly one of the following holds:*

(1) M is an FRI-monoid.
(2) There is a finitely generated submonoid $N \subseteq M$ and infinite subsets $S \subseteq \mathsf{R}(N)$, $S' \subseteq \mathsf{L}(N)$ such that (i) no two distinct elements of S have a right inverse in common and (ii) no two distinct elements of S' have a left inverse in common.

Proof. The conditions are clearly mutually exclusive. If M is not an FRI-monoid, it has a finitely generated submonoid N with infinite $\mathsf{R}(N)$. Distinguishing the cases of Lemma 2 for N yields the required sets. $\square$

We will employ a result by van Leeuwen [19] that generalizes Parikh's theorem. It states that semilinearity of all languages is preserved by building the algebraic extension of a language family.

Theorem 4 (van Leeuwen)**.** *Let $\mathcal{F}$ be a family of semilinear languages. Then every language that is algebraic over $\mathcal{F}$ is also semilinear.*

In light of the previous theorem, the following implies that the class of monoids M for which $\mathsf{VA}(M)$ contains only semilinear languages is closed under taking free products with amalgamation over a finite identified subgroup that contains the identity of each factor. In the case where the factors are residually finite groups, this was already shown in [15, Lemma 8] (however, for a more general operation than free products with amalgamation). The following also implies that if $\mathsf{VA}(M_i)$ contains only context-free languages for $i \in \{0,1\}$, then this is also true for $\mathsf{VA}(M_0 *_F M_1)$. This is due to the fact that clearly, the class of context-free languages is its own algebraic extension.

Theorem 5. *For each $i \in \{0,1\}$, let M_i be a finitely generated monoid and F be a subgroup that contains M_i's identity. Then every language in $\mathsf{VA}(M_0 *_F M_1)$ is algebraic over $\mathsf{VA}(M_0) \cup \mathsf{VA}(M_1)$.*

Proof. Since the algebraic extension of a full trio is again a full trio, it suffices to show that with respect to some generating set $S \subseteq M_0 *_F M_1$, the identity language of $M_0 *_F M_1$ is algebraic over $\mathsf{VA}(M_0) \cup \mathsf{VA}(M_1)$.

For $i \in \{0,1\}$, let $S_i \subseteq M_i$ be a finite generating set for M_i such that $F \subseteq S_i$. Furthermore, let X_i be an alphabet in bijection with S_i and let $\varphi_i : X_i^* \to M_i$ be the morphism extending this bijection. Moreover, let $Y_i \subseteq X_i$ be the subset with $\varphi_i(Y_i) = F$. Let $\psi_i : M_i \to M_0 *_F M_1$ be the canonical morphism. Since F is a subgroup of M_0 and M_1, ψ_0 and ψ_1 are injective (see e.g. [8, Theorem 8.6.1]). Let $X = X_0 \cup X_1$ and let $\varphi : X^* \to M_0 *_F M_1$ be the morphism extending $\psi_0\varphi_0$ and $\psi_1\varphi_1$. Then the identity language of $M_0 *_F M_1$ is $\varphi^{-1}(1)$ and we shall prove the theorem by showing that $\varphi^{-1}(1)$ is algebraic over $\mathsf{VA}(M_0) \cup \mathsf{VA}(M_1)$. We will make use of the following fact about free products with amalgamation of monoids with a finite identified subgroup. Let $s_1, \ldots, s_n, s'_1, \ldots, s'_m \in (X_0^* \setminus \varphi_0^{-1}(F)) \cup (X_1^* \setminus \varphi_1^{-1}(F))$, such that $s_j \in X_i^*$ if and only if $s_{j+1} \in X_{1-i}^*$ for $1 \leq j < n$, $i \in \{0,1\}$ and $s'_j \in X_i^*$ if and only if $s'_{j+1} \in X_{1-i}^*$ for $1 \leq j < m$, $i \in \{0,1\}$. Then the equality $\varphi(s_1 \cdots s_n) = \varphi(s'_1 \cdots s'_m)$ implies $n = m$. A stronger statement was shown in [14, Lemma 10]. We will refer to this as the *syllable property*.

For each $i \in \{0,1\}$ and $f \in F$, we define $L_{i,f} = \varphi_i^{-1}(f)$ and write y_f for the symbol in Y_i with $\varphi_i(y_f) = f^{-1}$. Then clearly $L_{i,1} \in \mathsf{VA}(M_i)$. Furthermore, since

$$L_{i,f} = \{w \in X_i^* \mid y_f w \in L_{i,1}\},$$

(here we again use that F is a group) we can obtain $L_{i,f}$ from $L_{i,1}$ by a rational transduction and hence $L_{i,f} \in \mathsf{VA}(M_i)$.

Let $\mathcal{F} = \mathsf{VA}(M_0) \cup \mathsf{VA}(M_1)$. Since for each $\mathcal{F}$-grammar G, it is clearly possible to construct an $\mathcal{F}$-grammar G' such that $L(G')$ consists of all sentential forms of G, it suffices to construct an $\mathcal{F}$-grammar $G = (N, T, P, S)$ with $N \cup T = X$ and

$S \Rightarrow_G^* w$ if and only if $\varphi(w) = 1$ for $w \in X^*$. We construct $G = (N, T, P, S)$ as follows. Let $N = Y_0 \cup Y_1$ and $T = (X_0 \cup X_1) \setminus (Y_0 \cup Y_1)$. As productions, we have $y \to L_{1-i,f}$ for each $y \in Y_i$ where $f = \varphi_i(y)$. Since $1 \in F$, we have an $e_i \in Y_i$ with $\varphi_i(e_i) = 1$. As the start symbol, we choose $S = e_0$. We claim that for $w \in X^*$, we have $S \Rightarrow_G^* w$ if and only if $\varphi(w) = 1$.

The "only if" is clear. Thus, let $w \in X^*$ with $\varphi(w) = 1$. We write $w = w_1 \cdots w_n$ such that $w_j \in X_0^* \cup X_1^*$ for all $1 \leq j \leq n$ such that $w_j \in X_i^*$ if and only if $w_{j+1} \in X_{1-i}^*$ for $i \in \{0,1\}$ and $1 \leq j < n$. We show by induction on n that $S \Rightarrow_G^* w$. For $n \leq 1$, we have $w \in X_i^*$ for some $i \in \{0,1\}$. Since $1 = \varphi(w) = \psi_i(\varphi_i(w))$ and ψ_i is injective, we have $\varphi_i(w) = 1$ and hence $w \in L_{i,1}$. This means $S = e_0 \Rightarrow_G w$ or $S = e_0 \Rightarrow_G e_1 \Rightarrow_G w$, depending on whether $i = 1$ or $i = 0$.

Now let $n \geq 2$. We claim that there is a $1 \leq j \leq n$ with $\varphi(w_j) \in F$. Indeed, if $\varphi(w_j) \notin F$ for all $1 \leq j \leq n$ and since $\varphi(w_1 \cdots w_n) = 1 = \varphi(\lambda)$, the syllable property implies $n = 0$, against our assumption. Hence, let $f = \varphi(w_j) \in F$. Furthermore, let $w_j \in X_i^*$ and choose $y \in Y_{1-i}$ so that $\varphi_{1-i}(y) = f$. Then $\psi_i(\varphi_i(w_j)) = \varphi(w_j) = f$ and the injectivity of ψ_i yields $\varphi_i(w_j) = f$. Hence, $w_j \in L_{i,f}$ and thus $w' = w_1 \cdots w_{j-1} y w_{j+1} \cdots w_n \Rightarrow_G w$. For w' the induction hypothesis holds, meaning $S \Rightarrow_G^* w'$ and thus $S \Rightarrow_G^* w$. □

4. Context-Freeness

In this section, we are concerned with the context-freeness of languages accepted by valence automata over graph products. The first lemma is a simple observation and we will not provide a proof. In the case of groups, it appeared in [6].

Lemma 6. *Let $\Gamma = (V, E)$ and $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ be a graph product. Then for each $v \in V$*

$$M \cong (M{\restriction}_{V \setminus \{v\}}) *_{M\restriction_{N(v)}} (M{\restriction}_{N(v)} \times M_v).$$

The following is a result by Lohrey and Sénizergues [13]. A finitely generated group is called *virtually free* if it has a free subgroup of finite index.

Theorem 7 (Lohrey, Sénizergues)**.** *Let G_v be a finitely generated non-trivial group for each $v \in V$. Then $\mathbb{M}(\Gamma, (G_v)_{v \in V})$ is virtually free if and only if*

1. *for each $v \in V$, G_v is virtually free,*
2. *if G_v and G_w are infinite and $v \neq w$, then $\{v, w\} \notin E$,*
3. *if G_v is infinite, G_u and G_w are finite and $\{v, u\}, \{v, w\} \in E$, then $\{u, w\} \in E$, and*
4. *the graph Γ is chordal.*

In order to prove that certain languages are not context-free, we will employ the following well-known Iteration Lemma by Ogden [16].

Lemma 8 (Ogden)**.** *For each context-free language L, there is an integer m such that for any word $z \in L$ and any choice of at least m distinct marked positions in z, there is a decomposition $z = uvwxy$ such that:*

1. *w contains at least one marked position.*
2. *Either u and v both contain marked positions, or x and y both contain marked positions.*
3. *vwx contains at most m marked positions.*
4. *$uv^iwx^iy \in L$ for every $i \geq 0$.*

Aside from Theorem 5, the following is the key tool to prove our result on context-freeness. We call a monoid M *context-free* if $\mathsf{VA}(M) \subseteq \mathsf{CF}$.

Lemma 9. *The direct product of monoids M_0 and M_1 is context-free if and only if for some $i \in \{0,1\}$, M_i is context-free and M_{1-i} is an FRI-monoid.*

Proof. Suppose M_i is context-free and M_{1-i} is an FRI-monoid. Then each language $L \in \mathsf{VA}(M_i \times M_{1-i})$ is contained in $\mathsf{VA}(M_i \times N)$ for some finitely generated submonoid N of M_{1-i}. Since M_{1-i} is an FRI-monoid, N has finitely many right-invertible elements and hence $\mathsf{J}(N)$ is a finite group. Since no element outside of $\mathsf{J}(N)$ can appear in a product yielding the identity, we may assume that $L \in \mathsf{VA}(M_i \times \mathsf{J}(N))$. This means, however, that L can be accepted by a valence automaton over M_i by keeping the right component of the storage monoid in the state of the automaton. Hence, $L \in \mathsf{VA}(M_i)$ is context-free.

Suppose $\mathsf{VA}(M_0 \times M_1) \subseteq \mathsf{CF}$. Then certainly $\mathsf{VA}(M_i) \subseteq \mathsf{CF}$ for each $i \in \{0,1\}$. This means we have to show that at least one of the monoids M_0 and M_1 is an FRI-monoid and thus, toward a contradiction, assume that none of them is. We provide two proofs for the fact that $\mathsf{VA}(M_0 \times M_1)$ contains non-context-free languages in this case. One is very short and the other is elementary in the sense that it does not invoke the fact that context-free groups are virtually free.

First proof. By Lemma 2, for each i, either $\mathsf{J}(M_i)$ is an infinite subgroup of M_i or M_i contains a copy of $\mathbb{B}$ as a submonoid. Since every infinite virtually free group contains an element of infinite order, we have that for each i, either (1) $\mathsf{J}(M_i)$ is an infinite group and hence contains a copy of $\mathbb{Z}$ or (2) M_i contains a copy of $\mathbb{B}$. In any case, $\mathsf{VA}(M_0 \times M_1)$ contains the language $\{a^n b^m c^n d^m \mid n, m \geq 0\}$, which is not context-free.

Second proof. By Lemma 3, for each i, there is a finitely generated submonoid $N_i \subseteq M_i$ and infinite sets $S_0 \subseteq \mathsf{R}(N_0)$ and $S_1 \subseteq \mathsf{L}(N_1)$ such that the elements of S_0 have pairwise disjoint sets of right inverses in N_0 and the elements of S_1 have pairwise disjoint sets of left inverses in N_1. Let X_i be an alphabet large enough that we can find a surjective morphism $\varphi_i : X_i^* \to N_i$ for each $i \in \{0,1\}$. Furthermore, let $\#$ be a symbol with $\# \notin X_0 \cup X_1$. The language

$$L = \{r_0 \# r_1 \# s_0 \# s_1 \mid r_i, s_i \in X_i^*,\ \varphi_i(r_i s_i) = 1 \text{ for each } i \in \{0,1\}\}$$

is clearly contained in $\mathsf{VA}(M_0 \times M_1)$. We shall use the Iteration Lemma to show that L is not context-free. Suppose L is context-free and let m be the constant provided by Lemma 8. For each $a \in \mathsf{R}(N_0)$, let $\ell_0(a)$ be the minimal length of a word $w \in X_0^*$ with $a\varphi_0(w) = 1$. Furthermore, for $a \in \mathsf{L}(N_1)$, let $\ell_1(a)$ be the minimal length of a word $w \in X_1^*$ with $\varphi_1(w)a = 1$. The existence of the sets S_0 and S_1 guarantees that there are $a_0 \in \mathsf{R}(N_0)$ and $a_1 \in \mathsf{L}(N_1)$ such that $\ell_0(a_0) \geq m$ and $\ell_1(a_1) \geq m$. Choose $r_0 \in X_0^*$ and $s_1 \in X_1^*$ such that $\varphi_0(r_0) = a_0$ and $\varphi_1(s_1) = a_1$. Furthermore, let $r_1 \in X_1^*$ be a word of minimal length among those with $\varphi_1(r_1 s_1) = 1$ and let $s_0 \in X_0^*$ be a word of minimal length among those with $\varphi_0(r_0 s_0) = 1$. These choices guarantee $|r_1| \geq m$ and $|s_0| \geq m$. Moreover, the word $z = r_0 \# r_1 \# s_0 \# s_1$ is in L.

Let $z = uvwxy$ be the decomposition provided by the Iteration Lemma, where we choose the positions in the subword $r_1 \# s_0$ to be marked. In the following, we call r_0, r_1, s_0, s_1 the *segments* of the word z. Clearly, v and x cannot contain the symbol $\#$. Therefore, by Condition (2), at least one of the words v and x lies in one of the middle segments. By Condition (3), they have to lie in the same segment or in neighboring segments. Hence, we have two cases:

- If v or x lies in the segment r_1, none of them lies in s_1. Thus, by pumping with $i = 0$, we obtain a word $r'_0\#r'_1\#s'_0\#s_1 \in L$ with $|r'_1| < |r_1|$ and $\varphi_1(r'_1 s_1) = 1$, contradicting the choice of r_1.
- If v or x lies in the segment s_0, none of them lies in r_0. Thus, by pumping with $i = 0$, we obtain a word $r_0\#r'_1\#s'_0\#s'_1 \in L$ with $|s'_0| < |s_0|$ and $\varphi_1(r_0 s'_0) = 1$, contradicting the choice of s_0.

This proves that L is not context-free and hence the lemma. □

We are now ready to prove our main result on context-freeness. Since for a graph product $M = \mathbb{M}(\Gamma, (M_v)_{v\in V})$, there is a morphism $\varphi_v : M \to M_v$ for each $v \in V$ that restricts to the identity on M_v, we have $\mathsf{J}(M) \cap M_v = \mathsf{J}(M_v)$: While the inclusion "$\supseteq$" is true for any submonoid, given $b \in \mathsf{J}(M) \cap M_v$ with $abc = 1$, $a, c \in M$, we also have $\varphi_v(a)b\varphi_v(c) = \varphi_v(abc) = 1$ and hence $b \in \mathsf{J}(M_v)$. This means no element of $M_v \setminus \mathsf{J}(M_v)$ can appear in a product yielding the identity. In particular, removing a vertex v with $\mathsf{J}(M_v) = \{1\}$ will not change $\mathsf{VA}(M)$. Hence, our requirement that $\mathsf{J}(M_v) \neq \{1\}$ is not a serious restriction.

Theorem 10. *Let $\Gamma = (V, E)$ and let $\mathsf{J}(M_v) \neq \{1\}$ for any $v \in V$. $M = \mathbb{M}(\Gamma, (M_v)_{v\in V})$ is context-free if and only if*

(1) for each $v \in V$, M_v is context-free,
(2) if M_v and M_w are not FRI-monoids and $v \neq w$, then $\{v, w\} \notin E$,
(3) if M_v is not an FRI-monoid, M_u and M_w are FRI-monoids and $\{v, u\}, \{v, w\} \in E$, then $\{u, w\} \in E$, and
(4) the graph Γ is chordal.

Proof. First, we show that conditions (1)–(4) are necessary. For (1), this is immediate and for (2), this follows from Lemma 9. If (3) is violated then for some $u, v, w \in V$, $M_v \times (M_u * M_w)$ is a submonoid of M such that M_u and M_w are FRI-monoids and M_v is not. Since M_u and M_w contain non-trivial (finite) subgroups, $M_u * M_w$ contains an infinite group and is thus not an FRI-monoid, meaning $M_v \times (M_u * M_w)$ is not context-free by Lemma 9.

Suppose (4) is violated for context-free M. By (2) and (3), any induced cycle of length at least four involves only vertices with FRI-monoids. Each of these, however, contains a non-trivial finite subgroup. This means M contains an induced cycle graph product of non-trivial finite groups, which is not virtually free by Theorem 7 and hence has a non-context-free identity language.

In order to prove the other direction, we note that $\mathsf{VA}(M) \subseteq \mathsf{CF}$ follows if $\mathsf{VA}(M') \subseteq \mathsf{CF}$ for every finitely generated submonoid $M' \subseteq M$. Since every such submonoid is contained in a graph product $N = \mathbb{M}(\Gamma, (N_v)_{v\in V})$ where each N_v is a finitely generated submonoid of M_v, it suffices to show that for such graph products, we have $\mathsf{VA}(N) \subseteq \mathsf{CF}$. This means whenever M_v is an FRI-monoid, N_v has finitely many right-invertible elements. Moreover, since $N_v \cap \mathsf{J}(N) = \mathsf{J}(N_v)$, no element of $N_v \setminus \mathsf{J}(N_v)$ can appear in a product yielding the identity. Hence, if N_v is generated by $S \subseteq N_v$, replacing N_v by the submonoid generated by $S \cap \mathsf{J}(N_v)$ does not change the identity languages of the graph product. Thus, we assume that each N_v is generated by a finite subset of $\mathsf{J}(N_v)$. Therefore, whenever M_v is an FRI-monoid, N_v is a finite group.

We first establish sufficiency in the case that M_v is an FRI-monoid for every $v \in V$ and proceed by induction on $|V|$. This means that N_v is a finite group for every $v \in V$. Since Γ is chordal, there is a $v \in V$ whose neighborhood is a clique.

This means $N{\restriction}_{N(v)}$ is a finite group and hence $N{\restriction}_{N(v)} \times N_v$ context-free by Lemma 9. Since $N{\restriction}_{V\setminus\{v\}}$ is context-free by induction, Theorem 5 and Lemma 6 imply that N is context-free.

To complete the proof, suppose there are n vertices $v \in V$ for which M_v is not an FRI-monoid. We proceed by induction on n. The case $n = 0$ is treated above. Choose $v \in V$ such that M_v is not an FRI-monoid. For each $u \in N(v)$, M_u is an FRI-monoid by condition (2), and hence N_u a finite group. Furthermore, condition (3) guarantees that $N(v)$ is a clique and hence $N{\restriction}_{N(v)}$ is a finite group. As above, Theorem 5 and Lemma 6 imply that N is context-free. □

Corollary 11. *Let* $\Gamma = (V, E)$*. Then* $\mathsf{VA}(\mathbb{M}(\Gamma, (M_v)_{v\in V})) \subseteq \mathsf{CF}$ *if and only if*

1. *for each* $v \in V$*,* $\mathsf{VA}(M_v) \subseteq \mathsf{CF}$*,*
2. *if* $\mathsf{REG} \subsetneq \mathsf{VA}(M_v)$ *and* $\mathsf{REG} \subsetneq \mathsf{VA}(M_w)$ *and* $v \neq w$*, then* $\{v, w\} \notin E$*,*
3. *if* $\mathsf{REG} \subsetneq \mathsf{VA}(M_v)$*,* $\mathsf{VA}(M_u) = \mathsf{VA}(M_w) = \mathsf{REG}$ *and* $\{v, u\} \in E$ *and* $\{v, w\} \in E$*, then* $\{u, w\} \in E$*, and*
4. *the graph* Γ *is chordal.*

5. Semilinearity

A well-known theorem by Chomsky and Schützenberger [2] was re-proved and phrased in terms of valence automata in the following way by Kambites [10].

Theorem 12. $\mathsf{VA}(\mathbb{Z} * \mathbb{Z}) = \mathsf{CF}$*.*

The next lemma can be shown using standard methods of formal language theory. See [15, 22] for a proof.

Lemma 13. *Let* M *be a monoid such that all languages in* $\mathsf{VA}(M)$ *are semilinear. Then every languages in* $\mathsf{VA}(M \times \mathbb{Z})$ *is semilinear.*

By a simple product construction, one can show the following.

Lemma 14. *If* $\mathsf{VA}(N_i) \subseteq \mathsf{VA}(M_i)$ *for* $i = 0, 1$*, then* $\mathsf{VA}(N_0 \times N_1) \subseteq \mathsf{VA}(M_0 \times M_1)$*.*

Lemma 15. $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$ *contains a non-semilinear language.*

Proof. $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$ is the class of languages accepted by partially blind two-counter machines [22]. Greibach [7] and, independently, Jantzen [9] have shown that such machines can accept the language $L_1 = \{wc^n \mid w \in \{0,1\}^*,\ n \leq \mathrm{bin}(w)\}$, where $\mathrm{bin}(w)$ denotes the number obtained by interpreting w as a base 2 representation: $\mathrm{bin}(w1) = 2 \cdot \mathrm{bin}(w) + 1$, $\mathrm{bin}(w0) = 2 \cdot \mathrm{bin}(w)$, $\mathrm{bin}(\lambda) = 0$. This means $L_1 \cap \{1\}\{0, c\}^* = \{10^n c^m \mid m \leq 2^n\}$ is also in $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$, which is clearly not semilinear. □

The next result also appears in [22], where, however, it was not made explicit that the undecidable language is unary.

Lemma 16. *If* Γ*'s underlying loop-free graph contains* P_4 *as an induced subgraph, then* $\mathsf{VA}(\mathbb{M}\Gamma)$ *contains an undecidable unary language.*

Proof. Let $\Gamma = (V, E)$ and $\mathring{\Gamma}$ be the graph obtained from Γ by adding a loop to every unlooped vertex. For notational reasons, we assume that the vertex set of $\mathring{\Gamma}$ is $\mathring{V} = \{\mathring{v} \mid v \in V\}$. Recall that $\mathbb{M}\Gamma$ is defined as $\mathbb{M}(\Gamma^-, (M_v)_{v\in V})$, where M_v is $\mathbb{Z}$ or $\mathbb{B}$, depending on whether v is looped or not. In the following, we write a_v and $\bar{a}_v$ for M_v's positive and negative generator, respectively. Lohrey and Steinberg

[15] show that there are rational sets $\mathring{R}, \mathring{S} \subseteq \mathbb{M}\mathring{\Gamma}$ over positive generators such that for a certain $\mathring{u} \in \mathring{V}$, given $n \in \mathbb{N}$, it is undecidable whether $1 \in a_{\mathring{u}}^n \mathring{R}\mathring{S}^{-1}$. Note that the morphism $\varphi : \mathbb{M}\Gamma \to \mathbb{M}\mathring{\Gamma}$ with $\varphi(a_v) = a_{\mathring{v}}$ and $\varphi(\bar{a}_v) = \bar{a}_{\mathring{v}}$ induces an isomorphism between the submonoids generated by positive generators and between the submonoids generated by the negative generators. Thus, we find rational sets $R, S \subseteq \mathbb{M}\Gamma$ over positive generators with $\varphi(R) = \mathring{R}$ and $\varphi(S) = \mathring{S}$.

If w is a word over positive generators in $\mathbb{M}\Gamma$, $w = a_1 \cdots a_n$, then we let $\bar{w} = \bar{a}_n \cdots \bar{a}_1$. This is well-defined, for if $a_1 \cdots a_n = b_1 \cdots b_m$, for positive generators $a_1, \dots, a_n, b_1, \dots, b_m$ then $\varphi(a_1 \cdots a_n) = \varphi(b_1 \cdots b_m)$ and thus $\varphi(\bar{a}_n \cdots \bar{a}_1) = \varphi(a_1 \cdots a_n)^{-1} = \varphi(b_1 \cdots b_m)^{-1} = \varphi(\bar{b}_m \cdots \bar{b}_1)$ and therefore $\bar{a}_n \cdots \bar{a}_1 = \bar{b}_m \cdots \bar{b}_1$. Note that $w\bar{w} = 1$ for every word w over positive generators. With this definition, the set $\bar{S} = \{\bar{s} \mid s \in S\}$ is also rational. We claim that for a word $w \in \mathbb{M}\Gamma$ over positive generators, $1 \in wR\bar{S}$ if and only if $1 \in \varphi(w)\mathring{R}\mathring{S}^{-1}$.

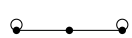
If $1 \in \varphi(w)\mathring{R}\mathring{S}^{-1}$, there are $\mathring{r} \in \mathring{R}$, $\mathring{s} \in \mathring{S}$ with $1 = \varphi(w)\mathring{r}\mathring{s}^{-1}$ and hence $\mathring{s} = \varphi(w)\mathring{r}$. Thus, we can find $s \in S$ and $r \in R$ with $\varphi(s) = \varphi(w)\varphi(r)$. The injectivity of φ on words over positive generators yields $s = wr$ and thus $1 = wr\bar{s}$. Hence $1 \in wR\bar{S}$.

If $1 \in wR\bar{S}$, we have $1 = wr\bar{s}$ for some $r \in R$ and $s \in S$. This implies $1 = \varphi(w)\varphi(r)\varphi(s)^{-1}$ and since $\varphi(r) \in \mathring{R}$ and $\varphi(s)^{-1} \in \mathring{S}^{-1}$, we have $1 \in \varphi(w)\mathring{R}\mathring{S}^{-1}$.

Thus, given $n \in \mathbb{N}$, it is undecidable whether $1 \in a_u^n R\bar{S}$. Now, we construct a valence automaton over $\mathbb{M}\Gamma$ that reads a word a^n while multiplying a_u in the storage for each input symbol and then nondeterministically multiplies an element from R and then an element from $\bar{S}$. It accepts if and only if $1 \in a_u^n R\bar{S}$. Therefore, the automaton accepts an undecidable unary language. $\square$

We are now in a position to show the first main result of this section. Note that the first condition of the following theorem is similar to conditions (2) and (3) in Theorem 10 (and 7): instead of FRI-monoids (finite groups) we have looped vertices and instead of non-FRI-monoids (infinite groups), we have unlooped vertices.

Theorem 17. *All languages in* $\mathsf{VA}(\mathbb{M}\Gamma)$ *are semilinear if and only if*

(1) Γ *contains neither* *nor*  *as an induced subgraph and*
(2) Γ*'s underlying loop-free graph contains neither* C_4 *nor* P_4 *as an induced subgraph.*

Proof. Let $\Gamma = (V, E)$. Suppose conditions (1) and (2) hold. We proceed by induction on $|V|$. (2) implies that Γ's underlying loop-free graph is a transitive forest. If Γ is not connected, then $\mathbb{M}\Gamma$ is a free product of graph products $\mathbb{M}\Gamma_1$ and $\mathbb{M}\Gamma_2$, for which $\mathsf{VA}(\mathbb{M}\Gamma_i)$ contains only semilinear languages by induction. Hence, by Theorems 4 and 5, every language in $\mathsf{VA}(\mathbb{M}\Gamma)$ is semilinear. If Γ is connected, there is a vertex $v \in V$ that is adjacent to every vertex other than itself. We distinguish two cases.

- If v is a looped vertex, then $\mathsf{VA}(\mathbb{M}\Gamma) = \mathsf{VA}(\mathbb{Z} \times \mathbb{M}(\Gamma \setminus \{v\}))$, which contains only semilinear languages by induction and Lemma 13.
- If v is an unlooped vertex, then by (1), $V \setminus \{v\}$ induces a clique of looped vertices. Thus, $\mathbb{M}\Gamma \cong \mathbb{B} \times \mathbb{Z}^{|V|-1}$, meaning $\mathsf{VA}(\mathbb{M}\Gamma)$ contains only semilinear languages by Lemma 13.

We shall now prove the other direction. If Γ contains as an induced subgraph, then $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$ is included in $\mathsf{VA}(\mathbb{M}\Gamma)$ and the former contains a non-semilinear language by Lemma 15. If Γ contains , then $\mathbb{M}\Gamma$ contains a copy of $\mathbb{B} \times (\mathbb{Z} * \mathbb{Z})$ as a submonoid. By Theorem 12, we have $\mathsf{VA}(\mathbb{B}) \subseteq \mathsf{VA}(\mathbb{Z} * \mathbb{Z})$ and hence Lemma 14 implies $\mathsf{VA}(\mathbb{B} \times \mathbb{B}) \subseteq \mathsf{VA}(\mathbb{B} \times (\mathbb{Z} * \mathbb{Z}))$.

Suppose Γ's underlying loop-free graph contains C_4 as an induced subgraph. Since we have already shown that the presence of or as an induced subgraph guarantees a non-semilinear language in $\mathsf{VA}(\mathbb{M}\Gamma)$, we may assume that all four participating vertices are looped. Hence, $\mathbb{M}\Gamma$ contains a copy of $(\mathbb{Z}*\mathbb{Z})\times(\mathbb{Z}*\mathbb{Z})$. By Theorem 12 and Lemma 14, this means $\mathsf{VA}(\mathbb{B} \times \mathbb{B}) \subseteq \mathsf{VA}(\mathbb{M}\Gamma)$. Thus, $\mathsf{VA}(\mathbb{M}\Gamma)$ contains a non-semilinear language. Finally, if Γ's underlying loop-free graph contains P_4 as an induced subgraph, Lemma 16 provides the existence of an undecidable unary language in $\mathsf{VA}(\mathbb{M}\Gamma)$. Since such a language cannot be semilinear, the lemma is proven. $\square$

5.1. **Torsion groups.** A *torsion group* is a group G in which for each $g \in G$, there is a $k \in \mathbb{N} \setminus \{0\}$ with $g^k = 1$. In this subsection, we show that for torsion groups G, all languages in $\mathsf{VA}(G)$ are semilinear. The key ingredient in our proof is showing that a certain set of multisets is upward closed with respect to a well-quasi-ordering. A *well-quasi-ordering on A* is a reflexive transitive relation $\leq$ on A such that for every infinite sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in A$, there are indices $i < j$ with $a_i \leq a_j$. We call a subset $B \subseteq A$ *upward closed* if $a \in B$ and $a \leq b$ imply $b \in B$. A basic observation about well-quasi-ordered sets states that for each upward closed set $B \subseteq A$, the set of its minimal elements is finite and B is the set of those $a \in A$ with $m \leq a$ for some minimal $m \in B$ (see [12]).

Given multisets $\alpha, \beta \in X^{\oplus}$ and $k \in \mathbb{N}$, we write $\alpha \equiv_k \beta$ if $\alpha(x) \equiv \beta(x) \pmod{k}$ for each $x \in X$. Furthermore, we write $\alpha \leq_k \beta$ if $\alpha \leq \beta$ and $\alpha \equiv_k \beta$. Clearly, $\leq_k$ is a well-quasi-ordering on $X^{\oplus}$: Since $\equiv_k$ has finite index in $X^{\oplus}$, we find in any infinite sequence $\alpha_1, \alpha_2, \ldots \in X^{\oplus}$ an infinite subsequence $\alpha'_1, \alpha'_2, \ldots \in X^{\oplus}$ of $\equiv_k$-equivalent multisets. Furthermore, $\leq$ is well-known to be a well-quasi-ordering [3] and yields indices $i < j$ with $\alpha'_i \leq \alpha'_j$ and hence $\alpha'_i \leq_k \alpha'_j$.

If $S \subseteq X^{\oplus}$ is upward closed with respect to $\leq_k$, we also say S is *k-upward-closed*. The observation above means in particular that every k-upward-closed set is semilinear.

Theorem 18. *For every torsion group G, the languages in $\mathsf{VA}(G)$ are semilinear.*

Proof. Let G be a torsion group and K be accepted by the valence automaton $A = (Q, X, G, E, q_0, F)$. We regard the finite set E as an alphabet and define the automaton $\hat{A} = (Q, E, G, \hat{E}, q_0, F)$ such that $\hat{E} = \{(p, (p, w, g, q), g, q) \mid (p, w, g, q) \in E\}$. Let $\hat{K} = L(\hat{A})$. Clearly, in order to prove Theorem 18, it suffices to show that $\hat{K}$ is semilinear.

For a word $w \in E^*$, $w = (p_1, x_1, g_1, q_1) \cdots (p_n, x_n, g_n, q_n)$, we write $\sigma(w)$ for the set $\{p_i, q_i \mid 1 \leq i \leq n\}$. w is called a *p, q-computation* if $p_1 = p$, $q_n = q$, and $q_i = p_{i+1}$ for $1 \leq i < n$. A q, q-computation is also called a *q-loop*. Moreover, a q-loop w is called *simple* if $q_i \neq q_j$ for $i \neq j$.

For each subset $S \subseteq Q$, let F_S be the set of all words $w \in E^*$ with $\sigma(w) = S$ and for which there is a $q \in F$ such that w is a q_0, q-computation and $|w| \leq |Q| \cdot (2^{|Q|}+1)$. Furthermore, let $L_S \subseteq E^*$ consist of all $w \in E^*$ such that w is a simple q-loop for some $q \in S$ and $\sigma(w) \subseteq S$. Note that L_S is finite, which allows us to define the alphabet Y_S so as to be in bijection with L_S. Let $\varphi : Y_S \to L_S$ be this bijection and let $\tilde{\varphi} : Y_S^{\oplus} \to E^{\oplus}$ be the morphism satisfying $\tilde{\varphi}(y) = \Psi(\varphi(y))$ for $y \in Y_S$.

For p, q-computations $v, w \in E^*$, we write $v \vdash w$ if $\sigma(v) = \sigma(w)$ and $w = rst$ such that r is a p, q'-computation, s is a simple q'-loop, t is a q', q-computation, and $v = rt$. Moreover, let $\preceq$ be the reflexive transitive closure of $\vdash$. In other words, $v \preceq w$ means that w can be obtained from v by inserting simple q-loops for states $q \in Q$ without increasing the set of visited states. For each $v \in F_S$, we define

$$U_v = \{\mu \in Y_S^{\oplus} \mid \exists w \in \hat{K} : v \preceq w,\ \Psi(w) = \Psi(v) + \tilde{\varphi}(\mu)\}$$

(note that there is only one $S \subseteq Q$ with $v \in F_S$). We claim that

$$(*) \qquad \Psi(\hat{K}) = \bigcup_{S \subseteq Q} \bigcup_{v \in F_S} \Psi(v) + \tilde{\varphi}(U_v).$$

The inclusion "$\supseteq$" holds by definition. For the other direction, we show by induction on n that for any $q_f \in F$ and any q_0, q_f-computation $w \in E^*$, $|w| = n$, there is a $v \in F_S$ for $S = \sigma(w)$ and a $\mu \in Y_S^{\oplus}$ with $v \preceq w$ and $\Psi(w) = \Psi(v) + \tilde{\varphi}(\mu)$. If $|w| \leq |Q| \cdot (2^{|Q|} + 1)$, this is satisfied by $v = w$ and $\mu = 0$. Therefore, assume $|w| > |Q| \cdot (2^{|Q|} + 1)$ and write $w = (p_1, x_1, g_1, q_1) \cdots (p_n, x_n, g_n, q_n)$. Since $n = |w| > |Q| \cdot (2^{|Q|} + 1)$, there is a $q \in Q$ that appears more than $2^{|Q|} + 1$ times in the sequence $q_1, \ldots, q_n$. Hence, we can write

$$w = w_0(p'_1, x'_1, g'_1, q)w_1 \cdots (p'_m, x'_m, g'_m, q)w_m$$

with $m > 2^{|Q|}+1$. Observe that for each $1 \leq i < m$, the word $w_i(p'_{i+1}, x'_{i+1}, g'_{i+1}, q)$ is a q-loop. Since $m - 1 > 2^{|Q|}$, there are indices $1 \leq i < j < m$ with

$$\sigma(w_i(p'_{i+1}, x'_{i+1}, g'_{i+1}, q)) = \sigma(w_j(p'_{j+1}, x'_{j+1}, g'_{j+1}, q)).$$

Furthermore, we can find a simple q-loop ℓ as a subword of $w_i(p'_{i+1}, x'_{i+1}, g'_{i+1}, q)$. This means for the word $w' \in E^*$, which is obtained from w by removing ℓ, we have $\sigma(w') = \sigma(w)$ and thus $w' \vdash w$. Moreover, with $S = \sigma(w)$ and $\varphi(y) = \ell$, $y \in Y_S$, we have $\Psi(w) = \Psi(w') + \tilde{\varphi}(y)$. Finally, since $|w'| < |w|$, the induction hypothesis guarantees a $v \in F_S$ and a $\mu \in Y_S^{\oplus}$ with $v \preceq w'$ and $\Psi(w') = \Psi(v) + \tilde{\varphi}(\mu)$. Then we have $v \preceq w$ and $\Psi(w) = \Psi(v) + \tilde{\varphi}(\mu + y)$ and the induction is complete. In order to prove "$\subseteq$" of $(*)$, suppose $w \in \hat{K}$. Since w is a q_0, q_f-computation for some $q_f \in F$, we can find the above $v \in F_S$, $S = \sigma(w)$, and $\mu \in Y_S^{\oplus}$ with $v \preceq w$ and $\Psi(w) = \Psi(v) + \tilde{\varphi}(\mu)$. This means $\mu \in U_v$ and hence $\Psi(w)$ is contained in the right hand side of $(*)$. This proves $(*)$.

By $(*)$ and since F_S is finite for each $S \subseteq Q$, it suffices to show that U_v is semilinear for each $v \in F_S$ and $S \subseteq Q$. Let $\gamma : E^* \to G$ be the morphism with $\gamma((p, x, g, q)) = g$ for $(p, x, g, q) \in E$. Since G is a torsion group, the finiteness of L_S permits us to choose a $k \in \mathbb{N}$ such that $\gamma(\ell)^k = 1$ for any $\ell \in L_S$. We claim that U_v is k-upward-closed. It suffices to show that for $\mu \in U_v$, we also have $\mu + k \cdot y \in U_v$ for any $y \in Y_S$. Hence, let $\mu \in U_v$ with $w \in \hat{K}$ such that $v \preceq w$ and $\Psi(w) = \Psi(v) + \tilde{\varphi}(\mu)$ and let $\mu' = \mu + k \cdot y$. Let $\ell = \varphi(y) \in L_S$ be a simple q-loop. Then $q \in S$ and since $\sigma(w) = \sigma(v) = S$, we can write $w = r(q_1, x_1, g_1, q)s$, $r, s \in E^*$. The fact that $w \in \hat{K}$ means in particular $\gamma(w) = 1$. Therefore, the word $w' = r(q_1, x_1, g_1, q)\ell^k s$ is a q_0, q_f-computation for some $q_f \in F$ and satisfies $\gamma(w') = 1$ since $\gamma(\ell)^k = 1$. This means $w' \in \hat{K}$ and $\Psi(w') = \Psi(w) + k \cdot \Psi(\ell) = \Psi(v) + \tilde{\varphi}(\mu + k \cdot y)$. We also have $\sigma(\ell) \subseteq S$ and hence $v \preceq w \preceq w'$. Therefore, $\mu' = \mu + k \cdot y \in U_v$. This proves U_v to be k-upward-closed and thus semilinear. $\square$

Render [18] has shown that for every monoid M, the class $\mathsf{VA}(M)$ either (1) coincides with the regular languages, (2) contains the blind one-counter languages, (3) contains the partially blind one-counter languages, or (4) consists of those accepted by valence automata over an infinite torsion group (which is not locally finite). Hence, we obtain the following.

Corollary 19. *For each monoid M, at least one of the following holds:*

(1) $\mathsf{VA}(M)$ *contains only semilinear languages.*
(2) $\mathsf{VA}(M)$ *contains the languages of blind one-counter automata.*
(3) $\mathsf{VA}(M)$ *contains the languages of partially blind one-counter automata.*

Since there are torsion groups with an undecidable word problem [1], we have:

Corollary 20. *There is a group G with an undecidable word problem such that all languages in* $\mathsf{VA}(G)$ *are semilinear.*

As another application, we can show that the one-sided Dyck language is not accepted by any valence automaton over $G \times \mathbb{Z}^n$, where G is a torsion group and $n \in \mathbb{N}$.

Corollary 21. *For torsion groups G and $n \in \mathbb{N}$, we have $D_1 \notin \mathsf{VA}(G \times \mathbb{Z}^n)$.*

Proof. First, observe that $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$ is not contained in $\mathsf{VA}(G \times \mathbb{Z}^n)$, since the former contains a non-semilinear language by Lemma 15 and the latter contains only semilinear ones by Theorem 18 and Lemma 13.

If D_1 were contained in $\mathsf{VA}(G \times \mathbb{Z}^n)$, then $\mathsf{VA}(\mathbb{B}) \subseteq \mathsf{VA}(G \times \mathbb{Z}^n)$, since D_1 is an identity language of $\mathbb{B}$. This means that $\mathsf{VA}(\mathbb{B} \times \mathbb{B})$ is contained in the class of languages accepted by valence automata over $(G \times \mathbb{Z}^n) \times (G \times \mathbb{Z}^n)$. The latter group, however, is isomorphic to $G^2 \times \mathbb{Z}^{2n}$, contradicting our observation above. $\square$

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