Valence automata as a generalization of automata with storage

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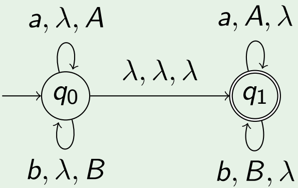
ALFA 2013

Example (Pushdown automaton)



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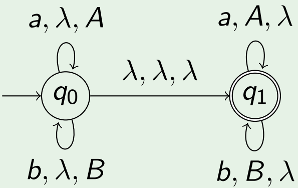

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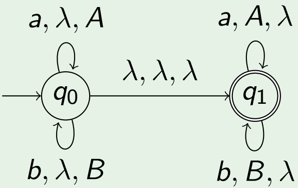

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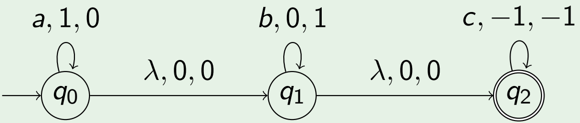

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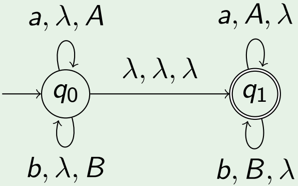

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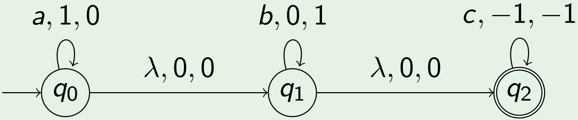
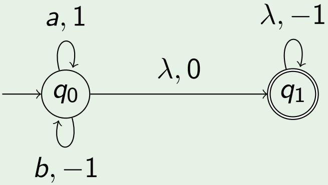


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



$$L = \{w \in \{a,b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for any prefix } p \text{ of } w\}$$

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

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Each storage mechanism consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \to S$

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Observation

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \mathrm{id}$$

Definition

A *monoid* is

- a set M together with
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- a neutral element $1 \in M$ ($a1 = 1a = a$ for any $a \in M$).

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Storage mechanisms as monoids

- Let S be a set of states and $\alpha_1, \ldots, \alpha_n : S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid M.
- The identity map is the neutral element of M.
- M is a decription of the storage mechanism.

Valence automata

Common generalization: Valence Automata

Valence automaton over M:

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Language class

VA(M) languages accepted by valence automata over M.

Valence automata

- Studied throughout the last decades
- In connection with valence grammars (Fernau, Stiebe)
- Expressive power for concrete monoids (Render, Kambites, Corson)
- As acceptors for word problems of groups (Gilman, Elder, Kambites, Ostheimer)

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detVA(M) languages accepted by deterministic valence automata over M.

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Theorem (Render 2010, Z. 2011)

The following statements are equivalent:

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Definition

A monoid with the above property is called *FRI-monoid*.

$$\mathsf{R}(M) = \{x \in M \mid \exists y \in M : xy = 1\} \quad \overline{\mathsf{R}}(x) = \{y \in M \mid xy = 1\}$$

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Lemma (Dichotomy)

For each monoid M, exactly one of the following holds:

1. $\mathrm{R}(M) = \mathrm{L}(M) = \mathrm{J}(M)$ *is a finite group.*
2. *There are infinite subsets $S \subseteq \mathrm{R}(M)$, $S' \subseteq \mathrm{L}(M)$ such that*
    - $\overline{\mathrm{R}}(s) \cap \overline{\mathrm{R}}(t) = \emptyset$ *for any $s, t \in S$ and*
    - $\overline{\mathrm{L}}(s) \cap \overline{\mathrm{L}}(t) = \emptyset$ *for any $s, t \in S'$.*

Definition (Graph products)

Let $\Gamma = (V, E)$ be a simple graph and M_v a monoid for each $v \in V$ with a presentation (A_v, R_v). Then the graph product $\mathbb{M}(\Gamma, (M_v)_{v \in V})$ is given by

$$A = \bigcup_{v \in V} A_v, \quad R = \bigcup_{v \in V} R_v \quad .$$

Intuition:

- $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ consists of sequences of elements in $\bigcup_v M_v$
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Specialization: Monoids defined by graphs

Notation

- $\mathbb{B}$: monoid for partially blind counter, $\mathbb{B} = \{a, \bar{a}\}^*/\{a\bar{a} = 1\}$.
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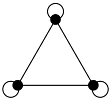
Monoids $\mathbb{M}\Gamma$

To each graph Γ, we associate the monoid $\mathbb{M}\Gamma$:

- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ is the corresponding graph product

Examples



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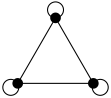

$\mathbb{Z}^3$

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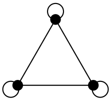

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Blind multicounter

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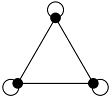
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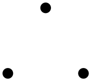

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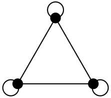

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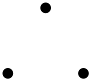

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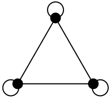

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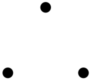

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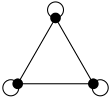

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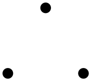

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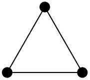

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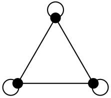

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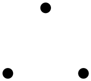

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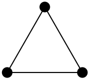

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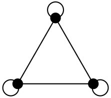

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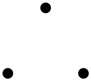

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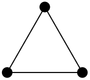

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Partially blind multicounter

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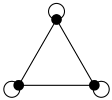

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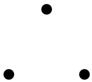

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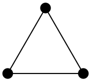

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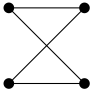

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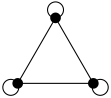

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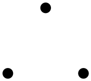

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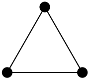

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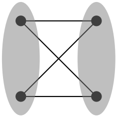

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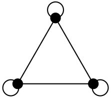

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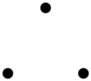

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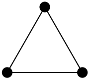

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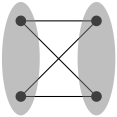

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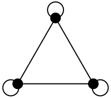

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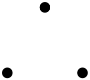

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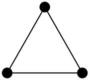

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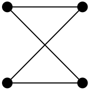

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Infinite tape (TM)

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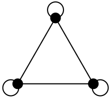

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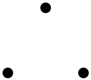

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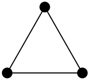

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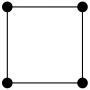

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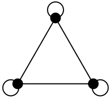

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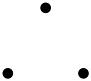

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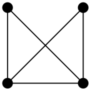

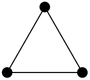

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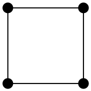

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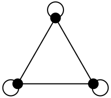

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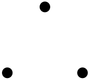

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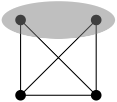

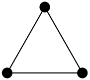

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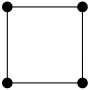

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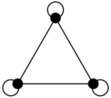

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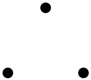

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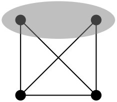

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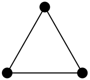

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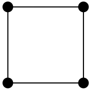

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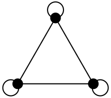

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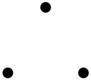

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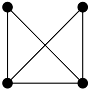

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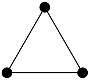
Pushdown + partially blind counters



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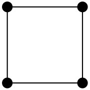

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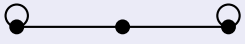
All languages in $\mathsf{VA}(\mathbb{M}\Gamma)$ *are semilinear if and only if*

1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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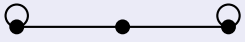
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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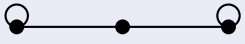
1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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- $\mathsf{VA}(\mathbb{B}) \subseteq \mathsf{CF}$
- $M \mapsto M \times \mathbb{Z}$, $(M, M') \mapsto M * M'$ preserve semilinearity

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A group G is called a *torsion group* if for every $g \in G$, there is a $k \in \mathbb{N} \setminus \{0\}$ with $g^k = 1$.

Theorem (Render 2010)

For every monoid M, at least one of the following holds:

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4. *the graph Γ is chordal.*

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Let $\Gamma = (V, E)$ *and let* $\mathsf{J}(M_v) \neq \{1\}$ *for any* $v \in V$. $M = \mathbb{M}(\Gamma, (M_v)_{v \in V})$ *is context-free iff*

1. *for each* $v \in V$, M_v *is context-free,*
2. *if* M_v *and* M_w *are not FRI-monoids and* $v \neq w$, *then* $\{v, w\} \notin E$,
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- For induced cycles of finite groups, relied on Lohrey and Sénizergues' result.

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