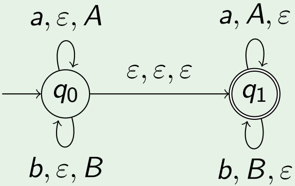
Of stacks (of stacks (...) with blind counters) with blind counters

Georg Zetzsche

Technische Universität Kaiserslautern

Algorithmics on Infinite State Systems 2014

Example (Pushdown automaton)



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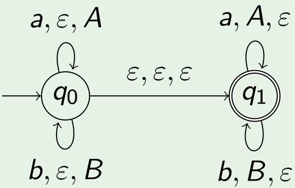

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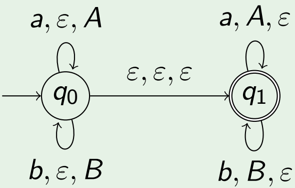

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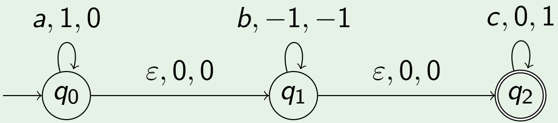

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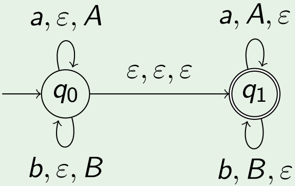

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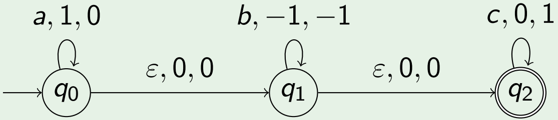
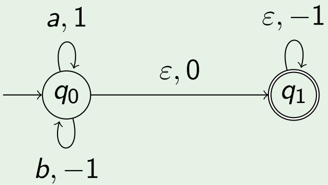


$$L = \{a^n b^n c^n \mid n \geqslant 0\}$$

Example (Partially blind counter automaton)



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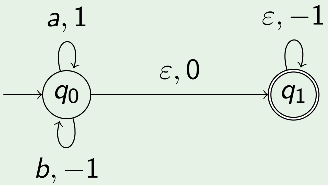

$$L = \{w \in \{a, b\}^* \quad | \quad |p|_a \geqslant |p|_b \text{ for each prefix } p \text{ of } w\}$$

Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

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Each storage mechanism consists of:

- States: set S of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n \colon S \to S$

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Pushdown automata	$S = \Gamma^*$	$\mathrm{push}_a\colon w \mapsto wa,\ a \in \Gamma$ $\mathrm{pop}_a\colon wa \mapsto w,\ a \in \Gamma$

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Observation

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \mathrm{id}$$

Definition

A *monoid* is

- a set M together with
- an associative binary operation $\cdot\colon M \times M \to M$ and
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Storage mechanisms as monoids

- Let S be a set of states and $\alpha_1, \ldots, \alpha_n \colon S \to S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid M.
- The identity map is the neutral element of M.
- M is a decription of the storage mechanism.

Valence automata

Common generalization: Valence Automata

Valence automaton over M:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.

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Language class

VA(M) languages accepted by valence automata over M.

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- For which do we have semilinearity of all languages?
- For which is the language class, for example, Boolean closed?
- For which can we decide, for example, emptiness?

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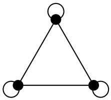
Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^*/\{a\bar{a} = \varepsilon\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $\mathbb{M}\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute

Examples



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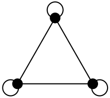
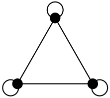


$\mathbb{Z}^3$

Examples



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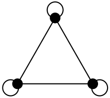
Blind counter

Examples



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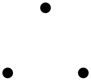

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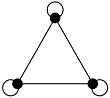

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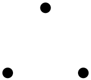

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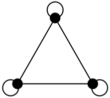

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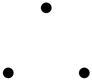

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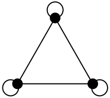

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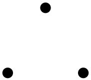

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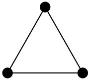

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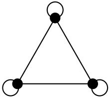

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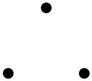

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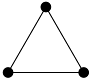

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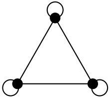

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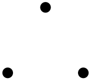

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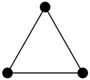

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Partially blind counter

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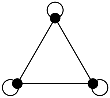

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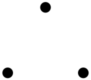

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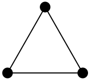

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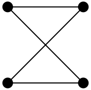

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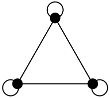

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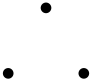

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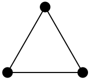

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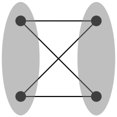

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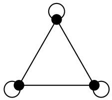

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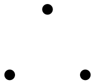

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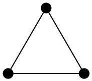

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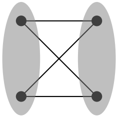

$(\mathbb{B} * \mathbb{B}) \times (\mathbb{B} * \mathbb{B})$

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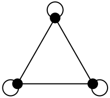

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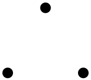

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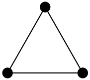

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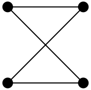

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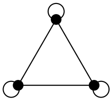
Infinite tape (TM)

Examples



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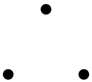

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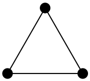
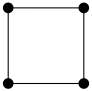


$\mathbb{B}^3$

Partially blind counter



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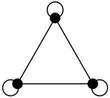
Infinite tape (TM)

Examples



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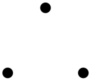

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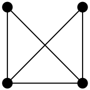

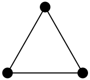

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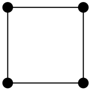

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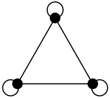

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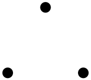

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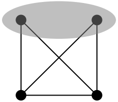

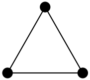

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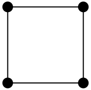

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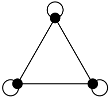

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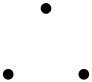

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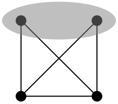

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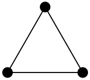

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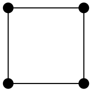

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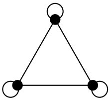

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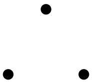

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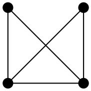

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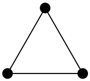
Pushdown + partially blind counters



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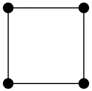

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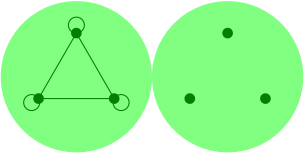
For which storage mechanisms can we avoid silent transitions?



Theorem (Z., ICALP 2013)

Let Γ be a graph such that

- *any two looped vertices are adjacent,*
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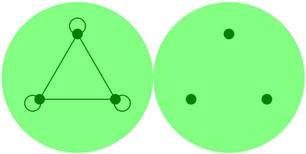

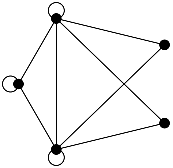

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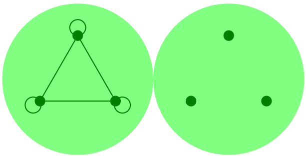

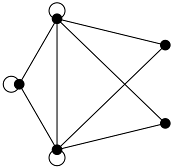

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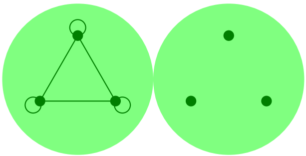

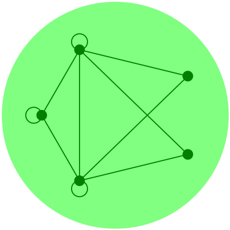

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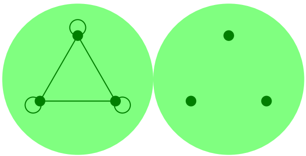

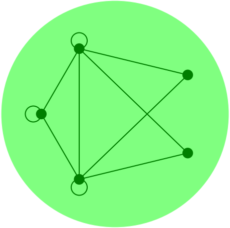

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Positive case

Definition (Stacked counters)

Let StCtr be the smallest class of monoids such that

- $1 \in \mathsf{StCtr}$
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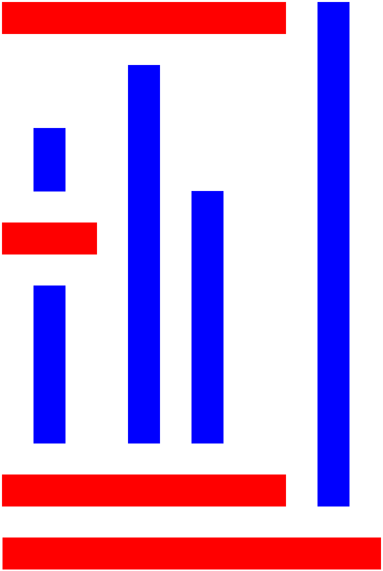

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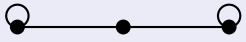
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
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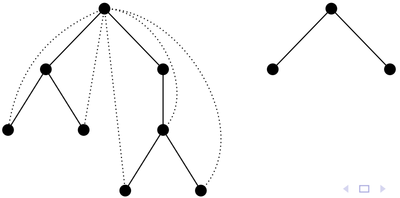

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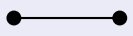
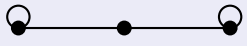
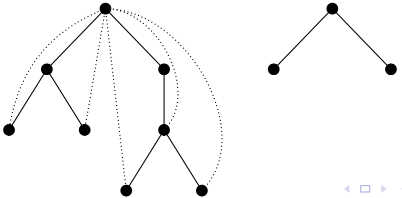
  1. Γ *contains neither*  *nor*  *as an induced subgraph and*
  2. Γ*, minus loops, is a transitive forest.*
- $\mathsf{VA}(\mathbb{M}\Gamma) \subseteq \mathsf{VA}(M)$ *for some* $M \in \mathsf{StCtr}$*. (NP-membership!)*



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Algebraic extensions

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- Nonterminals N, terminals T, start symbol $S \in N$
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Presburger constraints

For each language class $\mathcal{F}$, $\mathrm{SLI}(\mathcal{F})$ denotes the class of languages

$$h(L \cap \Psi^{-1}(S))$$

for some $L \in \mathcal{F}$, a homomorphism h and a semilinear set S.

A hierarchy of language classes

Hierarchy

$$\mathsf{F}_0 = \text{finite languages},$$

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Corollary

Stacked counter automata accept precisely the languages in F.

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For which systems can we compute $L{\downarrow}$?

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$u \preceq v$: u is obtained from v by arbitrarily deleting symbols

Theorem (Higman)

For every language $L \subseteq X^*$, the set $L{\downarrow} = \{u \in X^* \mid u \preceq v \text{ for some } v \in L\}$ is regular.

Applications

- $L{\downarrow}$ is observed through a lossy channel. Decidability for REG!
- Decide reversal boundedness. $\vartriangle$: up, $\triangledown$: down; $(\vartriangle\triangledown)^* \subseteq L{\downarrow}$?

Computability

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- for $\mathsf{Alg}(\mathcal{F})$ whenever computable for $\mathcal{F}$ (van Leeuwen 1978)

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- for $\mathsf{Alg}(\mathcal{F})$ whenever computable for $\mathcal{F}$ (van Leeuwen 1978)
- for Petri net languages (Habermehl, Meyer, Wimmel, ICALP 2010)

Computing the downward closure

Theorem

For stacked counter automata, downward closures can be computed.

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- Computability preserved by $\mathsf{Alg}(\cdot)$

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Parikh annotations

- New language in the same class
- Additional symbols encode decomposition of Parikh image into constant and period vectors
- Adding period vectors by inserting at designated positions

Parikh annotations

Example

$$L = (ab)^*(ca^* \cup db^*)$$

Parikh image: $\big(c + (a+b)^{\oplus} + a^{\oplus}\big) \cup \big(d + (a+b)^{\oplus} + b^{\oplus}\big)$.

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- Pumping lemma described by a language

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For each level F_i*, one can compute Parikh annotations in* F_i*.*

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Other applications of Parikh annotations include:

Theorem

For each $i \geqslant 0$*:* $\mathsf{F}_i \subsetneq \mathsf{G}_i \subsetneq \mathsf{F}_{i+1}$*.*

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- Decidability of questions for Büchi variants