Of stacks (of stacks (...) with blind counters) with blind counters

Georg Zetzsche

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Algorithmics on Infinite State Systems 2014
Example (Pushdown automaton)

\[
\begin{align*}
q_0 & \xrightarrow{a, \varepsilon, A} q_0 \\
q_0 & \xrightarrow{b, \varepsilon, B} q_1 \\
q_1 & \xrightarrow{a, A, \varepsilon} q_1 \\
q_1 & \xrightarrow{b, B, \varepsilon} q_1
\end{align*}
\]
Example (Pushdown automaton)

\[ L = \{ww^{rev} \mid w \in \{a, b\}^*\} \]
Example (Pushdown automaton)

\[
\begin{align*}
\delta(q_0, a) &= q_1, \\
\delta(q_0, b) &= q_1, \\
\delta(q_1, a) &= q_1, \\
\delta(q_1, b) &= q_1, \\
\end{align*}
\]

Language:
\[ L = \{ ww^\text{rev} \mid w \in \{a, b\}^* \} \]
Example (Pushdown automaton)

\[ L = \{ww^{\text{rev}} \mid w \in \{a, b\}^*\} \]

Example (Blind counter automaton)

\[ L = \{a^n b^n c^n \mid n \geq 0\} \]
Example (Partially blind counter automaton)

```
Example (Partially blind counter automaton)
```

![Diagram]

- States: $q_0$, $q_1$
- Edges:
  - $a, 1 \xrightarrow{} q_0$
  - $b, -1 \xrightarrow{} q_0$
  - $\varepsilon, 0 \xrightarrow{} q_0$
  - $\varepsilon, -1 \xrightarrow{} q_1$
  - $\varepsilon, 0 \xrightarrow{} q_1$

**For each prefix $p$ of $w$, $|p|_a = |p|_b$**
Example (Partially blind counter automaton)

\[ L = \{ w \in \{a, b\}^* \mid |p|_a \geq |p|_b \text{ for each prefix } p \text{ of } w \} \]
Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines
Automata models that extend finite automata by some storage mechanism:

- Pushdown automata
- Blind counter automata
- Partially blind counter automata
- Turing machines

Each storage mechanism consists of:

- States: set $S$ of states
- Operations: partial maps $\alpha_1, \ldots, \alpha_n : S \rightarrow S$
<table>
<thead>
<tr>
<th>Model</th>
<th>States</th>
<th>Operations</th>
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| Pushdown automata       | \( S = \Gamma^* \) | \( \text{push}_a: w \mapsto wa, \ a \in \Gamma \)  \
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**Observation**

Here, a sequence $\beta_1, \ldots, \beta_k$ of operations is valid if and only if

$$\beta_1 \circ \cdots \circ \beta_k = \text{id}$$
Definition

A monoid is

- a set $M$ together with
- an associative binary operation $\cdot : M \times M \rightarrow M$ and
- a neutral element $1 \in M$ ($a1 = 1a = a$ for any $a \in M$).
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Storage mechanisms as monoids
- Let $S$ be a set of states and $\alpha_1, \ldots, \alpha_n : S \rightarrow S$ partial maps.
- The set of all compositions of $\alpha_1, \ldots, \alpha_n$ is a monoid $M$.
- The identity map is the neutral element of $M$.
- $M$ is a description of the storage mechanism.
Common generalization: Valence Automata

Valence automaton over $M$:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.
Valence automata

Common generalization: Valence Automata

Valence automaton over $M$:

- Finite automaton with edges $p \xrightarrow{w|m} q$, $w \in \Sigma^*$, $m \in M$.
- Run $q_0 \xrightarrow{w_1|m_1} q_1 \xrightarrow{w_2|m_2} \cdots \xrightarrow{w_n|m_n} q_n$ is accepting for $w_1 \cdots w_n$ if
  - $q_0$ is the initial state,
  - $q_n$ is a final state, and
Valence automata

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Valence automata

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Language class

$\text{VA}(M)$ languages accepted by valence automata over $M$. 
Classical results can now be generalized:

**Questions**

- For which storage mechanisms can we avoid silent transitions?
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**Questions**

- For which storage mechanisms can we avoid silent transitions?
- For which do we have semilinearity of all languages?
- For which is the language class, for example, Boolean closed?
- For which can we decide, for example, emptiness?
Monoids defined by graphs

By graphs, we mean undirected graphs with loops allowed.
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$$X_\Gamma = \{a_v, \bar{a}_v \mid v \in V\}$$
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$$\cup \{xy = yx \mid x \in \{a_u, \bar{a}_u\}, y \in \{a_v, \bar{a}_v\}, \{u, v\} \in E\}$$
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$$M_\Gamma = X_\Gamma^*/R_\Gamma$$
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\[ M_\Gamma = X_\Gamma^*/R_\Gamma \]

Intuition

- $\mathbb{B}$: bicyclic monoid, $\mathbb{B} = \{a, \bar{a}\}^*/\{a\bar{a} = \varepsilon\}$.
- $\mathbb{Z}$: group of integers
- For each unlooped vertex, we have a copy of $\mathbb{B}$
- For each looped vertex, we have a copy of $\mathbb{Z}$
- $M_\Gamma$ consists of sequences of such elements
- An edge between vertices means that elements can commute
Examples

\[ 3 \]

Blind counter

\[ \hat{b} \]

\[ B \]

Pushdown

\[ 3 \]

Partially blind counter

\[ p \hat{b} \]

\[ q \hat{b} \]

Infinite tape (TM)

\[ p \hat{b} \]

\[ q \hat{b} \]

Pushdown + partially blind counters

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Valence Automata

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Examples

\[ \mathbb{Z}^3 \]
Examples

\[
\mathbb{Z}^3
\]

Blind counter
Examples

Blind counter
Examples

 Blind counter

\[ \mathbb{Z}^3 \]

\[ B \ast B \ast B \]
Examples

Blind counter

\( \mathbb{Z}^3 \)

Pushdown

\( \mathbb{B} \ast \mathbb{B} \ast \mathbb{B} \)
Examples

Blind counter

Pushdown

$\mathbb{Z}^3$

$B \ast B \ast B$
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

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Infinite tape (TM)

Pushdown + partially blind counters
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$B \cdot B \cdot B$

Partially blind counter

$B^3$
Examples

\[ \mathbb{Z}^3 \]

Blind counter

\[ \mathbb{B} \times \mathbb{B} \times \mathbb{B} \]

Pushdown

\[ \mathbb{B}^3 \]

Partially blind counter
Examples

Blind counter

Partially blind counter

Pushdown

\( \mathbb{Z}^3 \)

\( \mathbb{B}^3 \)
Examples

Blind counter

Partially blind counter

Pushdown

Infinite tape (TM)
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$B \ast B \ast B$

Partially blind counter

$B^3$

Infinite tape (TM)

$(B \ast B) \times (B \ast B)$
Examples

Blind counter

Pushdown

Partially blind counter

Infinite tape (TM)
Examples

Blind counter

Pushdown

Partially blind counter

Infinite tape (TM)
Examples

Blind counter

\[ \mathbb{Z}^3 \]

Pushdown

\[ \mathbb{B} \times \mathbb{B} \times \mathbb{B} \]

Partially blind counter

\[ \mathbb{B}^3 \]

Infinite tape (TM)

\[ (\mathbb{B} \times \mathbb{B}) \times (\mathbb{B} \times \mathbb{B}) \]
Examples

Blind counter

$\mathbb{Z}^3$

Pushdown

$B \times B \times B$

$(B \times B) \times B \times B$

Partially blind counter

$B^3$

Infinite tape (TM)

$(B \times B) \times (B \times B)$
Examples

Blind counter

Pushdown

Pushdown + partially blind counters

Partially blind counter

Infinite tape (TM)
Silent Transitions

A transition that reads no input is called *silent transition* or $\varepsilon$-transition.
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A transition that reads no input is called *silent transition* or \( \varepsilon \)-transition.

Important problem

- When can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.
Silent Transitions

A transition that reads no input is called *silent transition* or *ε-transition*.

Important problem

- When can silent transitions be eliminated?
- Without silent transitions, membership in NP.
- Elimination can be regarded as a precomputation.

Question

For which storage mechanisms can we avoid silent transitions?
Theorem (Z., ICALP 2013)

Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
- no two unlooped vertices are adjacent.
Theorem (Z., ICALP 2013)

Let $\Gamma$ be a graph such that

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Then the following conditions are equivalent:

- Silent transitions can be avoided over $M_\Gamma$,
- $\Gamma$ does not contain $P_{StCtr}$ as an induced subgraph.
Let $\Gamma$ be a graph such that

- any two looped vertices are adjacent,
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Then the following conditions are equivalent:

1. Silent transitions can be avoided over $\overline{M}\Gamma$.
2. $\Gamma$ does not contain $\bullet\longrightarrow\bullet\longrightarrow\bullet$ as an induced subgraph.
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Then the following conditions are equivalent:
- Silent transitions can be avoided over $\overline{\Gamma}$.
- $\Gamma$ does not contain $\cdot \rightarrow \cdot$ as an induced subgraph.
- $\overline{\Gamma} \in \text{StCtr}$
Positive case

**Definition (Stacked counters)**

Let \( \text{StCtr} \) be the smallest class of monoids such that

- \( 1 \in \text{StCtr} \)
- if \( M \in \text{StCtr} \), then \( M \times \mathbb{Z} \in \text{StCtr} \)
- if \( M \in \text{StCtr} \), then \( M * \mathbb{B} \in \text{StCtr} \)
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- If $M \in \text{StCtr}$, then $M \ast \mathbb{B} \in \text{StCtr}$

Interpretation of StCtr

StCtr corresponds to the class of storage mechanisms obtained by

- Adding a blind counter ($M \times \mathbb{Z}$):
  - States: $(c, z)$, $c$ an old state, $z \in \mathbb{Z}$.
  - Operations: old operations; increment, decrement for counter
Positive case

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- adding a blind counter ($M \times \mathbb{Z}$):
  - States: $(c, z)$, $c$ an old state, $z \in \mathbb{Z}$.
  - Operations: old operations; increment, decrement for counter
- building stacks ($M \star \mathbb{B}$)
  - States: sequences $\square c_1 \square c_2 \square \cdots \square c_n$, $c_i$ old states
  - Operations: push separator, pop if empty, manipulate topmost entry
Semilinearity
For which monoids $M$ are all languages in $\text{VA}(M)$ semilinear?

- Parikh’s Theorem: Pushdown automata
- Ibarra + Greibach: Blind counter automata
Semilinearity

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**Theorem (Buckheister, Z., MFCS 2013)**

Let $\Gamma$ be a graph. The following conditions are equivalent:

- All languages in $VA(M\Gamma)$ are semilinear.
- $\Gamma$ satisfies:
  1. $\Gamma$ contains neither $\bullet \longrightarrow \bullet$ nor $\bullet \longrightarrow \circ$ as an induced subgraph and
  2. $\Gamma$, minus loops, is a transitive forest.
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  1. $\Gamma$ contains neither $\bullet\longrightarrow\bullet$ nor $\bullet\longrightarrow\bigcirc$ as an induced subgraph and
  2. $\Gamma$, minus loops, is a transitive forest.
- $\text{VA}(\overline{M\Gamma}) \subseteq \text{VA}(M)$ for some $M \in \text{StCtr}$. (NP-membership!)
Expressiveness

Algebraic extensions

Let $F$ be a language class. An $F$-grammar $G$ consists of

- Nonterminals $N$, terminals $T$, start symbol $S \in N$
- Productions $A \rightarrow L$ with $L \subseteq (N \cup T)^*$, $L \in F$

Generated language: $\hat{w} = S^* \hat{w}$

Such languages are algebraic over $F$, class denoted $\text{Alg}_F$.

Presburger constraints

For each language class $F$, $\text{SLI}_F$ denotes the class of languages $h \hat{L} = \Psi$ for some $L \in F$, a homomorphism $h$ and a semilinear set $\Psi$. 
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- Such languages are algebraic over $\mathcal{F}$, class denoted $\text{Alg}(\mathcal{F})$.

Presburger constraints

For each language class $\mathcal{F}$, $\text{SLI}(\mathcal{F})$ denotes the class of languages

\[ h(L \cap \psi^{-1}(S)) \]

for some $L \in \mathcal{F}$, a homomorphism $h$ and a semilinear set $S$. 
A hierarchy of language classes

Hierarchy

\[ F_0 = \text{finite languages}, \]
\[ G_i = \text{Alg}(F_i), \quad F_{i+1} = \text{SLI}(G_i), \quad F = \bigcup_{i \geq 0} F_i. \]
A hierarchy of language classes

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Theorem

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)) \]
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**Theorem**

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)), \quad \bigcup_{i \geq 0} \text{VA}(M \times \mathbb{Z}^i) = \text{SLI}(\text{VA}(M)). \]
A hierarchy of language classes

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\[ G_i = \text{Alg}(F_i), \quad F_{i+1} = \text{SLI}(G_i), \quad F = \bigcup_{i \geq 0} F_i. \]

In particular: \( G_0 = \text{CF}. \)

\[ F_0 \subseteq G_0 \subseteq F_1 \subseteq G_1 \subseteq \cdots \subseteq F \]

Theorem

\[ \text{VA}(B \ast B \ast M) = \text{Alg}(\text{VA}(M)), \quad \bigcup_{i \geq 0} \text{VA}(M \times \mathbb{Z}^i) = \text{SLI}(\text{VA}(M)). \]

Corollary

Stacked counter automata accept precisely the languages in \( F \).
Downward closures

\( u \leq v: u \) is obtained from \( v \) by arbitrarily deleting symbols
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\( u \leq v \): \( u \) is obtained from \( v \) by arbitrarily deleting symbols

**Theorem (Higman)**

*For every language \( L \subseteq X^* \), the set \( L\downarrow = \{ u \in X^* \mid u \leq v \text{ for some } v \in L \} \) is regular.*
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- $L\downarrow$ is observed through a lossy channel.
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- \( L\downarrow \) is observed through a lossy channel. Decidability for \( \text{REG!} \)
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**Computability**

For which systems can we compute \( L \downarrow \)?
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- for \( \text{Alg}(F) \) whenever computable for \( F \) (van Leeuwen 1978)
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**Computability**

For which systems can we compute \( L\downarrow? \)

- for \( \text{Alg}(\mathcal{F}) \) whenever computable for \( \mathcal{F} \) (van Leeuwen 1978)
- for Petri net languages (Habermehl, Meyer, Wimmel, ICALP 2010)
Computing the downward closure

Theorem

For stacked counter automata, downward closures can be computed.
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- Computability preserved by Alg(·)
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- Computability preserved by Alg(·)
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Parikh annotations

- New language in the same class
- Additional symbols encode decomposition of Parikh image into constant and period vectors
- Adding period vectors by inserting at designated positions
Parikh annotations

Example

\[ L = (ab)^*(ca^* \cup db^*) \]

Parikh image: \( (c + (a + b)^\oplus + a^\oplus) \cup (d + (a + b)^\oplus + b^\oplus) \).
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\[ P = \{ p, q, r, s \}, \]
\[ C = \{ e, f \}, \]
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Parikh annotations

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\[ P = \{p, q, r, s\}, \]
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Makes Parikh decomposition accessible to transducers

Pumping lemma described by a language
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Theorem

For each level $F_i$, one can compute Parikh annotations in $F_i$. 

Other applications of Parikh annotations include:
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Computing downward closures

Recursively with respect to the hierarchy level:

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Conclusion

- Silent transitions avoidable, non-uniform membership in NP
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More classical results can be generalized:

Ongoing work

- Uniform word problem, connections to group theory
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